ARITHMETICAL INSTITUTIONS.

CONTAINING

A Compleat SYSTEM

OF

ARITHMETIC

Natural, Logarithmical, *and* Algebraical

IN ALL THEIR

BRANCHES:

WHEREBY

The Learner is led after an Easy and Familiar Manner from the very first Principles of this kind of LITERATURE to the State unto which it is brought at present:

TOGETHER

With many curious and useful IMPROVEMENTS never before made publick.

By the Rev. Mr. JOHN KIRKBY.

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TO THE

PRESIDENT,

Council, *and* Fellows,

OF THE

ROYAL SOCIETY

OF

LONDON

FOR THE

IMPROVING

OF

NATURAL KNOWLEDGE,

THIS

TREATISE

Is most humbly dedicated by the AUTHOR.

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The Application of Species Algorism *to* Indeterminate Problems.

PRÆ-

ERRATA.

IN 28. for *and Known*, r. *or Known*. In 45. l. last, for 9, r. 8. In. 86. l. last, r. $\frac{ax}{bx}$. In 160. l. last, r. *Reasons*. In. 198. l. 15, dele *M. M. L. L.* In 236. Con. 2, l. 7, add *BCE*, *CDE*. In 274. l. 2. r. *the greatest when*. In 287. l. 3, r. 274, 283. In. 290. p. 9, l. 6, r. 19470000. In 292. in the Table, r. 064, 084. In 311. for *Homogeneous*, r. *Pure*. In 380. p. 3. l. 4. for 10, r. 10*a*, and for *a*−20, r. −2*a*. In 383. l. 4. r. −*b*. In 417. Ex. 3, l. 1, r. −4. In 462. in the *Demonstration*, in the Register against Step 12, r. 10−11. In 463. l. last, for 38, r. 30. In 480. p. 20, l. 10. r. −4. In. 481. Ex. 1. l. 5, for −1*a*[2] r. + *a*[2]. In 498. Ex. 6. l. last, for 10, r. 4. In 619. r. *q*=7. In. 646. for *Demonstration*, r. *Definition*. In 647. l. 4. r. *to determine*. In. 670. l. 2. after Integers insert *upon certain Conditions given*. In. 691. Ex. 2. r. *d*3.

DIRECTIONS to the Book-binder

Observe the Numbers to the Left-hand in the Margin for the right placing of the Sheets.

PRÆCOGNOSCENDA.

1. A *Definition* is the Explication of such Words or Terms as want to be explained.

2. A *Partition*, or *Distinction* is the apt Distribution of Generals into their Particulars.

3. A *Proposition* is the pronouncing something true or false.

4. A *Demand* is a Proposition requiring something to be effected or granted.

5. A *Demonstration* is the Connexion of Arguments which are brought to prove the Truth or Falshood of a Proposition.

6. An *Effection* is the Method of answering or satisfying a Demand; and this, as well as *Demonstration*, is performed by *Analysis* or *Resolution*, when Things are, as it were, unravell'd into their first Principles; by *Synthesis* or *Composition* when the Proceeding is quite contrary.

7. An *Axiom* is a Proposition so easy in it self, that it needs no Demonstration.

8. A *Postulate* is a Demand so easy in it self, that it needs no Effection.

9. A *Theorem* is a Proposition, which requires a Demonstration.

10. A *Problem* is a Demand, which requires an Effection.

11. A *Corollary* is a Consequent to some preceding Truth.

12. A *Lemma* is a subsidiary Theorem, the Knowledge of which is necessary either for the Demonstration of some other Theorem, or the Effection of some Problem.

13. A *Scholium* is an accessary Remark, or an Observation for illustrating what is gone before, or about to follow.

14. An *Hypothesis* is the arbitrary supposition of such Signs and Terms, as are convenient and proper for treating what is in Hand.

15. One *Proposition* is said to be the *Converse* of another, when either supposes or implies the other.

16. *Q. E. D.* are the initial Letters of *Quod erat Demonstrandum*; i. e. which was to be demonstrated. *Q. E. E.* of *Quod erat efficiendum*; i. e. which was to be effected. *Ex. gr.* denotes *Exempli gratiâ*; i. e. for Example. *Vice versâ*, on the Contrary. *In.* Institution. *Pre.* Precept. *Con.* Conclusion. ∴ Therefore. *Ad Infinitum*, To Infinity.

ARITH-

ARITHMETICAL INSTITUTIONS.

PART I.

Of the first Elements of ARITHMETICK.

CHAP. I.

General DEFINITIONS.

DEFINITION I.

1. MATHEMATICKS is the Doctrine of *Quantity.*

DEFINITION II.

2. *Quantity* is whatever is the Subject of *Estimation,* or *Computation.*

DEFINITION III.

3. *Computation* is that Action of the Mind whereby Things are referred to *Unity.*

DEFINITION IV.

4. *Unity* is that whereby every thing is considered as *One.*

DEFINITION V.

5. *The same Units* are such as are apprehended under the same Notion; as the Gallons in a Bushel of Corn, or the Yards in a Web of Cloth. *Different Units* are such as are apprehended under different Notions, as the different Utensils in a Workman's Shop, or Letters in a Book.

DEFINITION VI.

6. Every Collection of Things taken as *Unity*, in respect of the Beings collected, is called a *Whole*, and the things collected are stiled its *Parts*; and if any one *Part* be assumed, the rest are stiled the *Complement of that Part to the Whole*. *Ex. gr.* If a Bushel of Wheat be considered as a *Whole* whose *Parts* are Eight Gallons, and if Five of those Gallons be assumed, then the remaining Three are the *Complement of that Part to the Whole*.

PARTITION I.

7. The Doctrine of Mathematicks is distinguished into *Abstract*, when it has no regard to Matter; and *Concrete*, when it has.

PARTITION II.

8. *Abstract Mathematicks* is two-fold, according as Quantity is the Subject: First, Of the Faculty of pure Intellection only, without any regard to the Images of Things which are impressed upon the Mind: Secondly, As it is the Subject of the imaginative Faculty; or as it respects the sensible Ideas under which material Beings are apprehended. The former is stiled *Arithmetick*, the latter *Geometry*. Hence

DEFINITION VII.

9. *Arithmetick* may be defined to be the Doctrine of Quantity, as it is the proper Subject of the pure Intellect. *Geometry*, the Doctrine of Quantity as it is represented to the Imagination; in the former respect, Quantity is the same with *Multitude*, in the latter with *Magnitude*.

COROLLARY I.

10. Hence *Arithmetick* is employed about Beings, both material, and immaterial: *Geometry*, only about Beings that are material.

DEFINITION VIII.

11. A *Multitude* is a Whole whose Parts are actually divided, and is therefore called *Discontinued Quantity*.

DEFINITION IX.

Definition IX.

12. A *Magnitude* is a Whole, whose Parts are only divisible in Power; and is therefore called *Continued Quantity*.

Definition X.

13. The Agreement of Things in Quantity, is called *Equality*, and their Disagreement, *Inequality*.

Definition XI.

14. Two Things are said to be *equal in Multitude*, when they are the same way referred to Unity in general.

Definition XII.

15. Two Things are said to be *equal in Magnitude*, when they are the same way referred to the same Unit.

Definition XIII.

16. Of two unequal Multitudes, that is said to be *More*, a Part of which is equal in Multitude with the Whole of the other; that *Fewer*, the Whole of which is equal in Multitude with a Part of the other.

Definition XIV.

17. Of two unequal Magnitudes that is said to be *Greater*, a Part of which is equal in Magnitude with the Whole of the other; that *Lesser*, the Whole of which is equal in Magnitude with a Part of the other.

Definition XV.

18. A Multitude is said to be *Homogeneous*, when it is of things of the same kind; *Heterogeneous*, when it is of things of different kinds.

Axiom I.

19. Every thing may be assumed as Unity.

Axiom II.

20. Every Quantity is equal to itself.

Axiom III.

21. Quantities which are equal to one and the same third, are equal to one another.

Axiom IV.

22. The Whole is more or greater than its Part.

AXIOM V.

23. The Whole is equal to all its Parts taken together.

POSTULATE I.

24. That one Quantity may be increased or diminished by another.

CHAP. II.

Of the Expression of QUANTITY.

PARTITION III.

25. THE Terms by which Quantity is expressed are *Species*, and *Number*.

DEFINITION XVI.

26. *Species* is that which expresses Quantity *indefinitely* and *universally.*

DEFINITION XVII.

27. *Number* is that which expresses Quantity *definitely* and *particularly.*

PARTITION IV.

28. *Species* are distinguished into *Given* and *Known*, and *Sought* or *Unknown.*

PARTITION V.

29. Both *Species* and *Number*, like the Quantities which they express, are divided into *Homogeneous*, and *Heterogeneous.*

HYPOTHESIS I.

30. The Species of Quantities are signified by the small Letters of the Alphabet, and sometimes by the Capital ones; Unknown Quantities by the Vowels, a, e, i, u, y; and Known ones by the Consonants, b, c, d, f, g, &c. according to Mr. *Harriot*. But according to others since him, Unknown Quantities are distinguished by the first Letters of the Alphabet, a, b, c, d, &c. and Known ones by the last, u, x, y, &c.

HYPOTHESIS II.

31. The Sign of *Equality* is $=$. The Sign of *Majority* is $\succ$, of *Minority* $\prec$. *Ex. gr.* $a=b$ denotes the Quantity represented under the Species a, to be equal to that known by the Species b; so $a \succ x$, signifies the Term a, to be greater than the Term x; and $x \prec a$, that x is lesser than a.

Scholium I.

32. The first Inventer of Species Arithmetick is said to have been *Francis Vieta*, a *Switzer*, who flourished about the Year 1590, but it was perfected by our Country Men *Harriot* and *Oughtred* who were Cotemporaries, *Anno Christi* 1600.

Chap. III.

Of Number *in general.*

Partition VI.

33. NUMBER is divided into *Integers* and *Fractions*.

Definition XVIII.

34. An *Integer* or *Whole Number* is that which is referred to Unity, as a Whole to a Part, as 1, 2, 3, 4, *&c.*

Definition XIX.

35. A *Fraction* or *broken Number* is that which is referred to Unity, as a Part to the Whole; as 1 Half, 2 Thirds, 1 Third, 3 Fourths.

Postulate II.

36. That no Integer can be assumed so great, but another may be assumed greater: Nor any Fraction so little, but another may be assumed less.

Definition XX.

37. An *Aliquot Part* is that which being some Number of Times repeated becomes equal to the Whole. An *Aliquant Part* is that which being repeated does either always exceed, or fall short of the Whole. *Ex. gr.* the Number 4 is an *Aliquot Part* of the Number 12, as being just 3 times contained in it: But the Number 5 is an *Aliquant Part* of 12, because if repeated twice it is less than 12; and if repeated thrice it becomes more.

Axiom VI.

38. Every lesser homogeneous Number is contained in a greater, either as an Aliquot, or an Aliquant Part.

Axiom VII.

39. Every Number is contained in it self once.

Axiom VIII.

40. Every Leſſer Number is contained in a Greater more than once.

Axiom IX.

41. The Greater any Number is in compariſon to another, the more equal Parts will it contain of that other.

Axiom X.

42. The nearer any leſſer Number approaches to being equal to a greater Number, the leſs often will it be contained in that greater Number.

Definition XXI.

43. The Compariſon between any two Homogeneous Numbers (the one being taken as a Part or Parts of the other) is ſtiled the *Ratio* of thoſe Numbers.

Definition XXII.

44. The *Ratio* of a greater Number to a leſſer (as of 8 to 2) is called a *Ratio* of *greater Inequality*, and the Ratio of a leſſer Number to a greater (as of 2 to 8) a *Ratio* of *leſſer Inequality*. And the former Term in every Ratio is called the *Antecedent*, the latter the *Conſeqnent*.

Axiom XI.

45. If two Numbers be encreaſed or decreaſed by like Parts of themſelves, their *Ratio*, both of *greater* and *leſſer Inequality*, continues ſtill the ſame. *Ex. gr.* If 21 and 12 be each encreaſed or decreaſed by their reſpective Thirds (*viz.* 7 and 4) the Sums 28 and 16 in the former Caſe, and the Remainders 14 and 9 in the latter, have the ſame *Ratio* to each other.

Definition XXIII.

46. A leſſer Number is ſaid to *meaſure* a greater, when it is an aliquot Part of that greater Number.

Definition XXIV.

47. A *Common Meaſure* of two or more Numbers is that which meaſures each. *Ex. gr.* 3 is a *Common Meaſure* of the Numbers 6, 9, 12, 15, 18, 21, *&c.*

Definition XXV.

48. *Commenſurate Numbers* are ſuch as have ſome one or more Common Meaſure beſides Unity, as the Numbers above.

Axiom XII.

49. If a Number meaſure all the Parts of another Number, it will alſo meaſure the Whole of that other Number.

Axiom XIII.

50. If a Number measure another Number, it will also measure all the Numbers, which that other Number measures.

Axiom XIV.

51. If a Number measuring any other Number do also measure a Part of that other Number, it will also measure the remaining Part.

Axiom XV.

52. If two Numbers are commensurate to a Third, or are commensurate to commensurate Numbers, they are commensurate to one another. And the same of incommensurate Numbers.

Definition XXVI.

53. An *Even Number* is that which is measured by 2.

Definition XXVII.

54. An *Odd Number* is one more than an even Number.

Definition XXVIII.

55. A *Prime* or *Incomposite Number* is that which no Number measures but Unity, as 3, 5, 7, 11, 13, 17, 19.

Definition XXIX.

56. A *Composite Number* is that which is measured by some one or more Numbers besides Unity, as 4, 6, 8, 9, 10, 12, 14, 15, *&c.*

Definition XXX.

57. A *Perfect Number* is that which is equal to all its aliquot Parts taken together; such, *ex. gr.* is the Number 6, whose aliquot Parts are, 1, 2, 3, and such again is 28, whose aliquot Parts are 1, 2, 4, 7, 14.

Chap. IV.

Of the Four First Rules of Arithmetical Invention.

Partition VII.

58. THE four first Rules of Arithmetical Invention are *Addition*, *Subtraction*, *Multiplication*, and *Division*.

Defini-

DEFINITION XXXI.

59. *Addition* is the Invention of a Number or Quantity called the *Sum* or *Aggregate*, by collecting together two or more given Homogeneous Quantities.

AXIOM XVI.

60. If equal Quantities be *added* to equal Quantities their Sums will be equal.

HYPOTHESIS III.

61. The Sign of *Addition* is $+$ *i. e. Plus* or *More*. *Ex. gr.* $6+2$ denotes the Sum of 6 and 2, or 6 more 2; and indefinitely $b+d$ denotes the *Sum* of the two given Quantities signified by the Species b and d.

DEFINITION XXXII.

62. *Subtraction* is the Invention of a Number or Quantity called the *Difference*, *Remainder*, or *Excess*, by taking a lesser given Quantity, called the *Subtrahend*, from a greater given Homogeneous Quantity called the *Minuend*.

AXIOM XVII.

63. If equal Quantities be *subtracted* from equal Quantities, the Remainders or Differences will be equal.

AXIOM XVIII.

64. If one equal Quantity be *subtracted* from another, the Remainder will be Nothing.

HYPOTHESIS IV.

65. The Sign of *Subtraction* is $-$ *i. e. Minus* or *Less*. *Ex. gr.* $6-2$ denotes the difference between 6 and 2, or 6 less 2; and indefinitely $b-d$ signifies the difference between the Species b, and the Species d, or b less d.

HYPOTHESIS V.

66. The Sign $\backsim$ denotes the difference of two Terms, without knowing which is greater, which lesser, *Ex. gr.* $a \backsim e$ serves indifferently for $a-e$ or $e-a$, according as a or e is greater.

COROLLARY II.

67. *Addition* and *Subtraction* are only of Homogeneous Terms. *Ex. gr.* $6s. + 2s. = 8s.$ *i. e.* 6 *Shillings* more 2 *Shillings* equal 8 *Shillings*: So $6d. - 2d. = 4d$: *i. e.* 6 *Denarii* or *Pence* less 2 *Pence* make 4 *Pence*. But $6s. + 2d.$ will neither make $8s.$ nor $8d.$ Nor will $6s. - 2d.$ make $4s.$ or $4d.$ but both must remain as they are, so long as they continue in different Denominations.

COROL-

Corollary III.

68. *Addition* is the Converse of *Subtraction*, and *Subtraction* of *Addition*. *Ex. gr.* if $a-b=d$, then because $b=b$ (In. 20.) $\because$ $a=d+b$ (In. 60.) And if $a=d+b$, then because $b=b$ (In. 20.) $\because$ $a-b=d$ (In. 63.)

Definition XXXIII.

69. *Multiplication* is the Invention of a Number or Quantity called the *Product*, by taking or adding a given Number called the *Multiplicand*, as often as there are Unites, or Parts of an Unite in another given Number called the *Multiplier*. *Ex. gr.* in the *Multiplication* of 6 by 2, 6 is the *Multiplicand*, 2 the *Multiplier*, and 12 the *Product*.

Definition XXXIV.

70. In *Multiplication* the *Multiplicand* and *Multiplier* are also called by the common name of *Factors*, or *Efficients*.

Axiom XIX.

71. If equal Quantities be multiplied by equal Quantities, the Products will be equal.

Hypothesis VI.

72. The Sign of *Multiplication* is $\times$ *i. e. Into*. *Ex. gr.* 6×2 denotes the *Product* of 6 multiplied by or into 2. And $b\times d$ or rather bd, like Letters in a Word, denotes the *Product* of b into d.

Hypothesis VII.

73. If one or both of the *Factors* do consist of more Terms than one connected by the Signs $+$ and $-$, a Line is to be drawn over all the Members in each. Thus the *Product* of $a+b-c$ multiplied into $x+z$ is signified by $\overline{a+b-c}\times\overline{x+z}$.

Definition XXXV.

74. Numbers multiplied into, or prefix'd to Species, are stiled *Coefficients*, and denote how often such Species are taken; thus, $5\,a$ is 5 times a: $14\,b$ is 14 times b, &c. And every Species without a *Coefficient* has Unity understood to be prefix'd to it; thus, a is $1\,a$, b is $1\,b$, &c.

Definition XXXVI.

75. A Product is said to be of as many *Dimensions* as it consists of literal Factors. *Ex. gr.* aa is a Product of two; $7abc$ of three; $xxxx$ of four, &c. Dimensions.

Definition XXXVII.

76. *Division* is the Invention of a Number or Quantity called the *Quotient*, which contains as many Units, as a given Quantity called the *Dividend* contains another given Quantity called the *Divisor*. *Ex. gr.* in the *Division* of 12 by 2. Here 12 is the *Dividend*, 2 the *Divisor*, and 6 the *Quotient*.

Axiom XX.

77. If equal Quantities be divided by equal Quantities, the Quotients will be equal.

Hypothesis VIII.

78. The Sign of *Division* is $\div$ *i. e. By*. *Ex. gr.* $6 \div 3$ denotes that 6 is to be divided by 3. Also here, as in *Multiplication*, when the *Divisor*, or the *Dividend*, or both are compound Quantities, a Line is to be drawn over all the Members in each; thus, the Division of $a+b-c$ by $x+z$ is denoted by $\overline{a+b-c} \div \overline{x+z}$.

Hypothesis IX.

79. Otherwise *Division* is signified by drawing a Line under the *Dividend*, and placing the *Divisor* beneath it; thus, $\frac{a+b-c}{x+z}$

Corollary IV.

80. The *Quotient* expresses the Ratio of the *Dividend* to the *Divisor* (In. 43.)

Axiom XXI.

81. Unity neither multiplies nor divides. *Ex. gr.* $1a = \frac{a}{1} = a$. $\because$ 2 is the least Integer that multiplies or divides.

Axiom XXII.

82. To multiply by an Integer encreases the Value of the Multiplicand, and to divide by an Integer decreases the Value of the Dividend.

Corollary V.

83. *Multiplication* is the Converse of *Division*, and *Division* of *Multiplication*; *i. e.* If in Multiplication the Product be divided by either Factor, the Quotient will be the other Factor. And in Division, if the Quotient be multiplied into the Divisor, the Product will be the Dividend. *Ex. gr.* If $\frac{a}{b} = d$ then

then becauſe $b=b$ (In. 20.) ∴ $a=bd$ (In. 71.) And if $a=bd$, then becauſe $b=b$ (In. 20.) ∴ $\frac{a}{b}=d$ (In. 77.)

Theorem I.

84. The Product of two Numbers or Quantities, (*Ex. gr.* $\overline{a+b}\times\overline{e+z}$ is equal to the Product of all the Parts of one multiplied (or drawn) into all the Parts of the other *i. e.* $\overline{a+b}\times\overline{e+z}=a\times\overline{e+z}+b\times\overline{e+z}$.

Demonſtration.

If equal Quantities are multiplied into equal Quantities, the Products are equal (In. 71.) But every Whole is equal to all its Parts taken together (In 23.) Therefore the Product made by multiplying two Wholes one into another $\overline{a+b}\times\overline{e+z}$ is equal to the Product made by multiplying all the Parts of one into all the Parts of the other; *i. e.* $\overline{a+b}\times\overline{e+z}=a\times\overline{e+z}+b\times\overline{e+z}$ or $e\times\overline{a+b}+z\times\overline{a+b}=ae+az+be+bz$. Q. E. D.

Corollary VI.

85. Therefore in Multiplication 'tis all one which of the Factors be the Multiplicand, and which the Multiplier.

Theorem II.

86. If a Dividend and Diviſor (*Ex. gr.* $\frac{a}{b}$) be both multiplied into (and conſequently both divided by) the ſame Quantity (*Ex. gr.* x) the Quotient will continue ſtill the ſame; *i. e.* $\frac{a}{b}=\frac{ax}{bx}$.

Demonſtration.

If a Diviſor and Dividend be encreaſed or diminiſhed by like Parts of themſelves, the Diviſor will be ſtill the ſame Way contained in the Dividend; *i. e.* their Ratio will ſtill be the ſame (In. 45.) But the Quotient is that which expreſſes the Ratio of the Dividend to the Diviſor (In. 80.) Therefore the Quotient is ſtill the ſame; *i. e.* $\frac{a}{b}=\frac{ax}{bx}$. Q. E. D.

Corollary VII.

87. Therefore wherever the ſame Term is found in both the Dividend, and the Diviſor, it is to be ſtruck out. Or in Numbers, whenever the Dividend and the Diviſor can both be divided by the ſame Number, it is to be done: Which is called a *bringing the Expreſſion to its loweſt Terms. Ex. gr.* $\frac{abx}{bza}=\frac{x}{z}$, $\frac{aa}{a}=\frac{a}{1}=a$ (In. 81.) $\frac{a}{a}=\frac{1}{1}=1$. (In 39.) And in Numbers $\frac{15}{3}=\frac{15\div3}{3\div3}=\frac{5}{1}=5$. $\frac{24\div6}{18\div6}=\frac{4}{3}$, &c.

Chap. V.

Of Fractions.

Definition XXXVIII.

88. If a Divisor be an Aliquant Part of the Dividend, then the Division of the Remainder by the same Divisor makes a *Fraction* (In. 35.) As when 14 is to be divided by 3, the Quotient is 4, with the third Part of 2, or two third Parts of 1, *i. e.* $\frac{14}{3} = 4 + \frac{2}{3}$. So $\frac{19}{5}$ or $19 \div 5 = \frac{15}{5} + \frac{4}{5} = 3 + \frac{4}{5}$ *i. e.* 3 and the 5th Part of 4, or 4 Fifths of 1.

Definition XXXIX.

89. The *Numerator* of a *Fraction* is the Number above, expressing how many Parts are taken in the Fraction. The *Denominator* is the Number beneath, which shews the Denomination of the Parts, or the Number of Parts into which the Unit is divided.

Partition VIII.

90. *Fractions* are divided into *Proper* and *Improper*.

Definition XL.

91. A *Proper Fraction* is that whose Numerator is less than the Denominator; as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{8}$, &c. An *Improper Fraction* is that whose Numerator is greater, as $\frac{7}{3}$, $\frac{8}{5}$, $\frac{12}{11}$, &c.

Partition IX.

92. *Fractions* are again divided into *Pure* and *Mixed*.

Definition XLI.

93. A *Pure Fraction* is that which is joined to no Integer, as $\frac{7}{8}$, $\frac{4}{5}$ &c. A *Mixed Number* or *Species* is that which is made up of an Integer, and a Pure Fraction, as $3 + \frac{1}{2}$; *i. e.* 3 and a half. $b + \frac{x}{z}$.

Scholium II.

94. In Mixed Numbers the Sign + which connects the Integer and Fraction is usually omitted. Thus $3 + \frac{1}{2}$ is writ $3\frac{1}{2}$, $15 + \frac{3}{4}$ is $15\frac{3}{4}$, but not so in Species.

Partition X.

Partition X.

95. *Pure Fractions* are divided into *Simple*, and *Compound*.

Definition XLII.

96. A *Simple Fraction* is that which is not divided into more, as the Examples above. A *Compound Fraction* is the Multiplication of Fractions, or the breaking a Fraction into more; as the Expressions $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{5}{7}$ of $\frac{3}{8}$ of $\frac{1}{9}$, $\frac{2}{5}$ of $\frac{11}{13}$, &c.

Partition XI.

97. *Simple Fractions* are either *Homogeneous*, or *Heterogeneous*.

Definition XLIII.

98. *Homogeneous Fractions* are such as have the same Denominator, or are referred to the same Unit. *Heterogeneous Fractions* are such as have different Denominators, or are referred to different Units.

Corollary VIII.

99. In Proper Fractions the Numerator is to the Denominator in a Ratio of lesser Inequality: In Improper Fractions in a Ratio of greater Inequality.

Corollary IX.

100. Every Integer may be looked upon as a Fraction whose Denominator is Unity; thus $5 = \frac{5}{1}$ $6 = \frac{6}{1}$, $b = \frac{b}{1}$, &c.

Corollary X.

101. When the Numerator and Denominator of a Fraction are the same, the Fraction is the same with Unity (In. 39.)

Corollary XI.

102. The greater the Denominator of a Fraction is in respect of its Numerator, the lesser is the Fraction; and *vice versâ*, the lesser the Denominator of a Fraction is in respect of its Numerator, the greater is the Fraction (In. 41, and 42.)

Corollary XII.

103. If both the Numerator and Denominator of a Fraction be multiplied (or divided) by the same Number or Quantity, the Fraction will still retain the value (In. 86, and 87.)

Corollary XIII.

104. If the Numerator of a Fraction be multiplied by any Number or Quantity, it is made so many times greater, as there are Units in that Number or Quantity; and if divided by it, so many times less (In. 102.)

Corollary XIV.

105. If the Denominator of a Fraction be multiplied by any Number or Quantity, it is made so many times lesser, as there are Units in that Quantity; and if divided by it, so many times greater (In. 102.)

Corollary XV.

106. Whence, to divide the Numerator of a Fraction by any Quantity, is all one as to multiply the Denominator of the same Fraction by that Quantity: And *vice versâ*, to multiply the Numerator of a Fraction by any Number or Quantity is all one as to divide the Denominator by that Number or or Quantity. *Ex. gr.* $\frac{a \times b}{aa} = \frac{b}{aa \div a} = \frac{b}{a}$.

Problem I.

107. To reduce Heterogeneous Fractions into Homogeneous ones retaining the same Value.

Effection.

Pre. 1. Multiply all the given Denominators together for a new and common Denominator.

2. Multiply each Numerator into all the Denominators, except its own, for new Numerators.

3. Subscribe the new and common Denominator under each of these new Numerators. Then I say that each Homogeneous Fraction, thus found, is equal to the respective given Heterogeneous one from whence its Numerator was formed. Q. E. E.

Example.

Let it be required to reduce the given Heterogeneous Fractions $\frac{b}{p}, \frac{c}{q}, \frac{d}{r}$ into Homogeneous ones of the same Value.

By *Pre.* 1. $p \times q \times r = pqr$ the common Denominator.

By *Pre.* 2. the new Numerators are bqr, cpr, and dpq.

Therefore by *Pre.* 3. the Homogeneous Fractions required are $\frac{bqr}{pqr}, \frac{cpr}{pqr}, \frac{dpq}{pqr}$

i. e. $\frac{bqr}{pqr} = \frac{b}{p}, \frac{cpr}{pqr} = \frac{c}{q}, \frac{dpq}{pqr} = \frac{d}{r}$ (In. 86, and 87.)

If

If $b=2$, $c=3$, $d=4$, $p=5$, $q=7$, $r=9$; then instead of $\frac{b}{p}, \frac{c}{q}, \frac{d}{r}$, we shall have $\frac{2}{5}, \frac{3}{7}, \frac{4}{9}$, which reduced by the foregoing Precepts become $\frac{2\times7\times9}{5\times7\times9}=\frac{126}{315}$; $\frac{3\times5\times9}{5\times7\times9}=\frac{135}{315}$; $\frac{4\times5\times7}{5\times7\times9}=\frac{140}{315}$.

Problem II.

108. To reduce a given Integer b into a Fraction of the same Value, whose Denominator shall be a given Quantity d.

Effection.

Multiply the given Integer b into the given Denominator d, and under the Product bd subscribe the Denominator d; then I say $\frac{bd}{d}=b$ (In. 86, and 87.) Q. E. E.

Problem III.

109. To reduce a mixed Quantity ($b+\frac{x}{z}$ or $b-\frac{x}{z}$) into an Improper Fraction of the same Value.

Effection.

Multiply the Integer b into the Denominator of the Fraction z, and under the Sum $bz+x$ in the former Case, or Difference $bz-x$ in the latter Case, subscribe the Denominator z; then I say $\frac{bz+x}{z}=b+\frac{x}{z}$, and $\frac{bz-x}{z}=b-\frac{x}{z}$. Q. E. E.

Demonstration.

The Quantity $b=\frac{bz}{z}$ (In. 86.) Therefore $\frac{bz}{z}+\frac{x}{z}=b+\frac{x}{z}$ (In. 60.) But $\frac{bz}{z}+\frac{x}{z}=\frac{bz+x}{z}$ or $\overline{bz+x}\div z$ (In. 88.) Therefore $\frac{bz+x}{z}=b+\frac{x}{z}$ (In. 21.) And after the same manner will it be proved that $\frac{bz-x}{z}$ or $\frac{bz}{z}-\frac{x}{z}$. $=b-\frac{x}{z}$. Q. E. D.

Problem IV.

110. To reduce an Improper Fraction $\frac{bz+x}{z}$ to its equivalent Integer or mixed Quantity.

Effection.

Divide the Numerator $bz+x$ by the Denominator z, and under the Remainder subscribe the Denominator. I say $b+\frac{x}{z}=\frac{bz+x}{z}$. Q. E. E.

Problem V.

PROBLEM V.

111. To add one given Fraction to another Fraction given.

Effection.

Pre. 1. If any of the given Fractions to be added together be a mixed Quantity, reduce it to an Improper Fraction (In. 109.) And if they are Heterogeneous reduce them to Homogeneous ones (In. 107.)

2. Add all the Numerators together, and under that Sum subscribe the common Denominator, and it is done.

Ex. gr. $\frac{b}{x}+\frac{c}{x}+\frac{d}{x}=\frac{b+c+d}{x}$. So $\frac{3}{4}+\frac{1}{4}=\frac{4}{4}=1$ (In. 39.) Again $2\frac{1}{2}+\frac{2}{3}=\frac{5}{2}+\frac{2}{3}$ (In. 109.) $=\frac{15}{6}+\frac{4}{6}$ (In. 107.) $=\frac{19}{6}=3\frac{1}{6}$ (In. 110.)

PROBLEM VI.

112. To subtract a given Fraction from another Fraction given.

Effection.

Pre. 1. If either of the given Fractions be mixed, reduce it to an Improper Fraction (In. 109.) And if they are Heterogeneous make them Homogeneous (In. 107.)

Pre. 2. Subtract the lesser Numerator from the greater, and under that Difference subscribe the common Denominator, and it is done.

Ex. gr. $\frac{b}{x}-\frac{c}{x}=\frac{b-c}{x}$, $\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$ (In. 87.) So $2\frac{1}{2}-\frac{2}{3}=\frac{5}{2}-\frac{2}{3}$ (In. 109.) $=\frac{15}{6}-\frac{4}{6}$ (In. 107.) $=\frac{11}{6}=1\frac{5}{6}$ (In. 110.)

PROBLEM VII.

113. To multiply a Fraction $\frac{b}{c}$ by a Fraction $\frac{d}{p}$ (In. 96.)

Effection.

Multiply all the Numerators together for a new Numerator, and all the Denominators together for a new Denominator, and it is done; I say, $\frac{b\times d}{c\times p}$ $=\frac{bd}{cp}$ the Product required.

Demonstration.

Put $\frac{b}{c}=x$, $\frac{d}{p}=z$, then will $xz=$ the required Product (In. 71.) But if $\frac{b}{c}=x$, then $b=cx$; if $\frac{d}{p}=z$, then $d=pz$ (In. 83.) And if $b=cx$, and $d=pz$, then $bd=cpxz$ (In. 71.) And lastly, if $bd=cpxz$, because $cp=cp$, therefore $\frac{bd}{cp}=xz$; *i. e.* $\frac{bd}{cp}$ is the Product required. Q. E. D.

If

If $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, $\frac{d}{e}$ be to be multiplied together, $\frac{a}{b} \times \frac{b}{c} = \frac{ab}{bc}$, $\frac{ab1}{bc} \times \frac{c}{d} = \frac{abc}{bcd}$, $\frac{abc}{bcd} \times \frac{d}{e} = \frac{abcd}{bcde}$, therefore $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} = \frac{abcd}{bcde} = \frac{a}{e}$ (In. 87.) In Numbers $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{24}{120} = \frac{1}{5}$ (In 87.) So $2 \times \frac{3}{4} = \frac{2}{1} \times \frac{3}{4}$ (In. 100.) $= \frac{6}{4} = \frac{3}{2}$ (In. 87.) $= 1\frac{1}{2}$ (In 110.)

Problem VIII.

114. To divide a Fraction $\frac{bd}{cp}$ by a Fraction $\frac{d}{p}$.

Effection.

Pre. 1. When the Numerator of the Dividend can be divided by the Numerator of the Divisor, and the Denominator of the Dividend by the Denominator of the Divisor; then the respective Quotients will be the Numerator and Denominator of the Quotient required; thus, $\frac{bd}{cp} \div \frac{d}{p} = \frac{b}{c}$, $\frac{3}{15} \div \frac{3}{5} = \frac{1}{3}$, $\frac{12}{18} \div \frac{6}{9} = \frac{2}{2} = 1$. But when this cannot be done, then

Pre 2. Multiply the Numerator of the Dividend into the Denominator of the Divisor for a new Numerator, and the other Numerator and Denominator together for a new Denominator, and it is done. I say $\frac{bd}{cp} \div \frac{d}{p} = \frac{bdp}{cdp} = \frac{b}{c}$ (In. 87.) So $\frac{2}{5} \div \frac{3}{4} = \frac{8}{15}$, $5 \div \frac{2}{3}$ or $\frac{5}{1} \div \frac{2}{3} = \frac{15}{2} = 7\frac{1}{2}$, $\frac{2}{3} \div 5$ or $\frac{2}{3} \div \frac{5}{1} = \frac{2}{15}$, &c.

Demonstration.

Since Division is the Converse of Multiplication (In. 83.) then because the Multiplication of Fractions is performed by Multiplying all the Numerators together, and all the Denominators together (In 113.) therefore the Division of one Fraction by another must be performed by dividing the Numerator of the Dividend by the Numerator of the Divisor, and the Denominator of the Dividend by the Denominator of the Divisor, according to Pre. 1. above: But because this is all one as to multiply the contrary Numbers together, as in Pre. 2. (In. 106.) therefore, &c. Q. E. D.

Corollary XVI.

115. Therefore to divide the Denominator of a Fraction by any Integer is the same as to multiply the Fraction by that Integer; and to divide the Numerator of a Fraction by an Integer is the same as to divide the Fraction by that Integer; thus, $\frac{b}{zz} \times z = \frac{b}{z}$, and $\frac{bb}{z} \div b = \frac{b}{z}$.

Corollary XVII.

116. Also it appears that to divide by a Fraction always encreases the Value of the Dividend, and to multiply by a Fraction decreases the Value of the Multiplicand, which is the Converse of Axiom 22, (In. 82.)

Corollary XVIII.

117. The Product made by two Integers will it self be an Integer; and the Product made of two proper Fractions will be a Fraction.

Corollary XIX.

118. That Number which being divided by an Integer makes an Integer in the Quotient is it self an Integer (In. 83.) And *vice versa* that Number, which being divided by a proper Fraction, makes a proper Fraction in the Quotient, is it self a proper Fraction.

Chap. VI.

Of the Greatest Common Measure *between two or more given Numbers or Quantities.*

Definition LXIV.

119. THE *Greatest Common Measure* between two given Numbers or Quantities, is that, which dividing both shall bring them to their *least* or *lowest Terms* (In. 87.)

Problem IX.

120. To find the Greatest Common Measure between two given Numbers, *Ex. gr.* b the greater, and c the lesser.

Effection.

Pre 1. Divide the greater Number b by the lesser c, and call the Remainder d.

Pre 2. Divide the first Divisor c by the first Remainder d, and call the second Remainder f.

Pre. 3. Divide the second Divisor d, by the second Remainder f, and call the third Remainder g.

Pre. 4. Continue so doing as long as any thing remains, and the last Divisor will be the greatest Common Measure sought. Q. E. E.

The

The Demonstration of this *Effection* is twofold.

First, to prove, that the last Divisor, how many soever there be thus found, does really *measure* the first given Numbers b and c.

Suppose *Ex. gr.* that the second Remainder f does divide the second Divisor d without a Remainder, so that the third Remainder $g=o$.

Con. 1. The Number d measures the Complement of f to c (Pre. 2.) and the Number f measures d by Supposition: But if a Number measures another Number, it also measures every Number which that other Number measures (In. 50.) therefore the Number f measures the Complement of f to c.

2. The Number f measures its Complement to c (Con. 1.) and it also measures it self (In. 39.) But if a Number measure all the Parts of another Number, it also measures the whole of that Number (In. 49.) therefore f measures c, the lesser given Number.

3. The Number c measures the Complement of d to b (Pre. 1.) and f measures c (Con. 2.) But if, &c. (In. 50.) therefore f measures the Complement of d to b.

4. The Number f measures the Complement of d to b (Con. 3.) and also measures d by Supposition. But if, &c. (In. 49.) therefore f also measures b the greater given Number.

It is plain therefore the last Divisor thus found, how many soever there be, will be a *Common Measure* to the given Numbers. It follows next to demonstrate that it is the *Greatest Common Measure*.

Suppose z for the *Greatest Common Measure* between the given Numbers b and c.

Con. 1. If z measures b and c (as by the Supposition) it must also measure what remains after the Division of b by c, or the Complement of all the c's, that b will contain, to b (In. 51.) But that Complement is d (Pre. 1.) Therefore z measures d.

2. If z measures c and d (Con. 1.) by like reason it must measure f and so on. But if z measure f it cannot possibly be greater than f. Therefore f is the greatest Common Measure between b and c. Q. E. D.

Corollary XX.

121. If the lesser given Number be an Aliquot Part of the greater, then is the lesser Number the *greatest Common Measure*.

Corollary XXI.

122. If the last Divisor be Unity, then are the given Numbers in their lowest Terms already.

Corollary XXII.

123. If a Number z measure any two Numbers b, and c, it will also mea- the *greatest Common Measure* between those two Numbers.

Corollary XXIII.

124. If a Number z which is the *greatest Common Measure* to any two other Numbers b and c do measure a third d, it will be the *greatest Common Measure* to all the three, b, c, d. For since z is the *greatest Common Measure* that can be to b and c, therefore no greater Number measuring d can measure both b and c.

Problem X.

135. To find the *greatest Common Measure* to any Number of Terms; *Ex. gr.* b, c, d, f.

Effection.

Pre. 1. Find the *greatest Common Measure* between the first Term b, and the second c, (In. 120.) which call p.

2. Find the *greatest common Measure* between p, and the third Term d, which call q.

3. Find the *greatest Common Measure* between q and the fourth Term f, which call r, and so on if Occasion require to a fifth, sixth, &c. Term at pleasure; and the last *Common Measure* thus found is the *greatest* to all the Terms for which it is found; consequently r is the *greatest Common Measure* to the four Terms b, c, d, f.

Demonstration.

Con. 1. The Number p measures b and c (Pre. 1.) and q measures p (Pre 2.) But, &c. (In. 50.) Therefore q measures b and c.

2. The Number r measures q (Pre. 3.) q measures b and c (Con. 1.) and q measures d (Pre. 2.) But, &c. (In. 50.) Therefore r measures b, c, d.

And since r measures f (Pre. 3.) Therefore r measures b, c, d, and f, which was the first Thing to be proved.

Secondly, Suppose z for the *greatest Common Measure* required.

Con. 1. The Number z measures b and c (by Supposition) and p is the *greatest Common Measure* between b and c (Pre. 1.) But if, &c. (In. 123.) Therefore z measures p.

2. The Number z measures p (Con. 1.) and d (by Supposition) and q is the *greatest Common Measure* between p and d (Pre. 2.) But, &c. (In. 123.) Therefore z measures q.

And after the same manner will it be proved that z measures r the *Common Measure* found, but if z measures r it cannot possibly be lesser than r, there-

therefore *r* is the *greatest Common Measure* between the given Numbers *b*, *c*, *d*, *f*. Q. E. D.

COROLLARY XXIV.

126, If a Number *z* measure ever so many Numbers, it will also measure the *greatest Common Measure* of those Numbers.

CHAP. VII.

Of the Powers of QUANTITIES.

DEFINITION XLV.

127. THE continued Multiplication of a Quantity into it self is termed *Involution*; and according to the Number of such Multiplications of the same Quantity, it is said to be raised to the *First*, *Second*, *Third*, *Fourth*, &c. *Power* or *Degree*; *Ex. gr.* If the Number or Quantity *a* be considered as multiplied into Unity (*i. e.* as 1 *a*) it is termed the *Root* or *First Power* of *a*; then that first Power multiplied into it self composes the *Square* or *second Power* (*aa*) : That *second Power* multiplied into the *Root a*, composes the *Cube* or *third Power (aaa)*: That *third Power* multiplied into the *Root a* composes the *Biquadrate* or *fourth Power (aaaa)* : That *fourth Power* multiplied into the *Root a* composes the *first Surfolid* or *fifth Power (aaaaa)*, &c. *ad infinitum.*

DEFINITION LXVI.

128. The Converse of *Involution*, *i. e.* the Resolution of the *Root* from the given *Power*, is called *Evolution*, or *Extraction.*

DEFINITION LXVII.

129. A *Scale* of *Powers* are all those in their Order, which are raised from the same *Root*, as, *a*, *aa*, *aaa*, *aaaa*, *aaaaa*, *aaaaaa*, *aaaaaaa*, &c.

PARTITION XII.

130. *Powers* are divided into *Homogeneous* and *Heterogeneous.*

DEFINITION XLVIII.

131. *Homogeneous Powers* are such as belong to the same *Scale*; as, *a*, *aa*, *aaa*, &c. *Heterogeneous Powers* belong to different *Scales*; as, *a*, *bb*, *cc*, *ddd*, &c.

PARTITION XIII.

132. *Heterogeneous Powers*, as well as their *Roots*, are either *Homologous*, or *Heterologous*.

DEFINITION XLIX.

133. *Homologous Powers* are such *Heterogeneous Powers* as are of the same Dimension; as, *aa* and *bb*, *ccc* and *ddd*, *xxxx* and *zzzz*, &c. all others are *Heterologous*.

DEFINITION L.

134. *Homologous Roots* are those of the same Name (*viz.* the *Square*, *Cube*, *Biquadrate*, &c. *Roots*) which are extracted from *Heterogeneous Quantities*.

DEFINITION LI.

135. *Homologous Roots* extracted from *Commensurate* Quantities are said to be *Commensurate in Power*.

PARTITION XIV.

136. *Powers*, as well as their *Roots*, are again distinguished into *Rational* and *Irrational*.

DEFINITION LII.

137. *Rational Powers* are such as have their *Roots* capable of being expressed in Numbers, as $4 = 2 \times 2$, $8 = 2 \times 2 \times 2$, $64 = 4 \times 4 \times 4$, &c. And such *Roots* are called *Rational Roots*.

DEFINITION LIII.

138. *Irrational Powers* are all Numbers, which either in reality have no Roots at all to be expressed in Numbers (as 2, 3, 5, 6, 7, 10, &c.) or at least no such as are required to be extracted from them: Of which kind, *Ex. gr.* are 4 taken as a *Cube*; 8 taken as a *Square*, or a *Biquadrate*; 9 taken as a *Surfolid*, or as a *sixth Power*, &c. And their *Roots* are stiled *Irrational Roots*, or *Surds*.

DEFINITION LIV.

139. Every Number, or Species, whose *Root* is required to be extracted, is called a *Resolvend*.

HYPOTHESIS X.

140. The *Powers* of Quantities are denoted (according to the Example of *Kepler*, and after him *Cartesius*) by Numbers called *Exponents*, which are placed at the Head of the Quantity to be involved to the right Hand: Thus, *a* is

a is a^1, aa is a^2, $aaa = a^3$, $aaaa = a^4$, &c. so $\overline{a+b}^5$ signifies the fifth *Power* of $\overline{a+b}$. Whence

Hypothesis XI.

141. The *Roots* of Quantities are denoted by *Fractional Exponents* set at the Head of the *Resolvend*, whose Numerators are Unity, and whose Denominators are the Exponent of the Power, which such *Resolvend* is taken for. Thus *Ex. gr.* the *Square Root* of a, or a^2 is $a^{\frac{1}{2}}$, its Cube Root $a^{\frac{1}{3}}$, its Biquadrate $a^{\frac{1}{4}}$, &c. and the *Root* of $\overline{a+b}$ taken as a *seventh Power* is $\overline{a+b}^{\frac{1}{7}}$.

Hypothesis XII.

142. The Species m is put universally to signify the *Exponents* of all *Powers*. Thus, a^m, b^m, c^m, denote indefinitely the *first*, *second*, *third*, *fourth*, &c. Powers of the Quantities, a, b, c. And $a^{\frac{1}{m}}$, $b^{\frac{1}{m}}$, $c^{\frac{1}{m}}$ denote the *Square*, *Cube*, *Biquadrate*, &c. Roots; according as m is put for 2, 3, 4, *&c.*

Hypothesis XIII.

143. The Sign $\circledcirc^m$ set after any Quantity denotes that such Quantity is to be *involved* to the m *Power* (*i. e.* to the *Square*, *Cube*, *Biquadrate*, &c. according to what m is put for.) And on the contrary ɯ^m signifies that the m *Root* is to be *extracted* from it; *Ex. gr.* $a \circledcirc^3$ signifies that a is to be involved into a *Cube*, and $a\,\text{ɯ}^3$ that its *Cube Root* is to be extracted.

Definition LV.

144. Quantities expressed by *Exponents* are called *Exponential Quantities.*

Definition LVI.

145. All *Powers* above the third, whose Exponents are prime Numbers, are stiled *Sursolids*: Thus a^5 is the first *Sursolid Power* of a, a^7 the *second Sursolid*, a^{11} the *third Sursolid*, &c.

Axiom XXIII.

146. Homologous Powers raised from equal Quantities are equal; and, *vice versâ*, Homologous Roots extracted from equal Quantities are equal.

Corollary XXV.

147. Homogeneous Powers are Heterogeneous Terms, and consequently, as such can neither be added nor subtracted.

Corollary XXVI.

148. Irrational Roots are always expressed by their Exponents.

Corollary XXVII.

COROLLARY XXVII.

149. Every Integer may be considered as a Power whose Exponent is Unity; *Ex. gr.* a is a^1, 5 is 5^1, 8 is 8^1, *&c.*

COROLLARY XXVIII.

150. Powers whose Exponents are Composit Numbers may have their Roots extracted, according to the Numbers of which their Exponents are composed. Thus the Biquadrate Root is extracted by extracting first the Square Root, and then the Square Root of that Square Root. Also the Root of the sixth Power is extracted by extracting first the Square Root, and then the Cube Root of that Square Root; or otherwise, by extracting first the Cube Root, and then the Square Root of that Cube Root. But in Practice it is always most convenient to extract the lower Root first; *i. e.* the Square Root before the Cube, the Cube before the Root of the fifth Power, *&c.*

COROLLARY XXIX.

151. An Exponential Quantity is involved to the second Power by doubling the Exponent; to the third Power by tripling it, to the fourth by quadrupling it, *&c.* *Ex. gr.* $a^1 \odot^2 = a^2$, $a^2 \odot^2 = a^{2\times2} = a^4$, $a^3 \odot^3 = a^{3\times3} = a^9$, $a^3 \odot^5 = a^{3\times5} = a^{15}$, $b^{\frac{1}{2}} \odot^2 = b^{\frac{1}{2}\times2} = b^1$ or b, $x^{\frac{3}{4}} \odot^2 = x^{\frac{6}{4}} = x^{\frac{3}{2}} = x^{1\frac{1}{2}} = \overline{xxx}^{\frac{1}{2}}$.

COROLLARY XXX.

152. An Exponential Quantity is evolved to the Square Root by halfing the Exponent, to the Cube by thirding it, to the Biquadrate Root by fourthing it, *&c.* *Ex. gr.* $a^6 \text{ɯ}^2 = a^{6\div2} = a^3$, $a^3 \text{ɯ}^3 = a^{3\div3} = a$, $a^{\frac{1}{2}} \text{ɯ}^2 = a^{\frac{1}{2}\div2} = a^{\frac{1}{4}}$, $a^2 \text{ɯ}^{\frac{1}{2}} = a^{2\div\frac{1}{2}} = a^4$, *&c.*

COROLLARY XXXI.

153. Homogeneous Powers are multiplied together by adding their Exponents, and divided by subtracting them; *Ex. gr.* $a \times a = a^{1+1} = a^2$, $b^2 \times b^3 = b^{2+3} = b^5$, $c^{\frac{2}{3}} \times c^{\frac{1}{2}} = c^{\frac{2}{3}+\frac{1}{2}} = c^{\frac{7}{6}}$ (In. 111.) $a^2 \div a^1 = a^{2-1} = a$, $b^5 \div b^3 = b^{5\ \ 3} = b^2$, $b^3 \div b^5 = b^{3-5} = b^{-2} = \frac{bbb}{bbbbb} = \frac{1}{bb}$ (In. 87.)

COROLLARY XXXII.

154. Whence the Quotients resulting from the Division of a lesser Homogeneous Power by a greater may be thus expressed; $\frac{aa}{aaaaaaa} = \frac{1}{aaaaa} = a^{2-7}$
$= a^{0-5}$

$= a^{0-5} = a^{-5}$, $\frac{a}{aaaaa} = \frac{1}{aaaa} = a^{1-5} = a^{-4}$, $\frac{a}{aaaa} = \frac{1}{aaa} = aa^{1-4} = a^{-3}$.
$\frac{a}{aaa} = \frac{1}{aa} = a^{1-3} = a^{-2}$, $\frac{a}{aa} = \frac{1}{a} = a^{1-2} = a^{-1}$, $\frac{a}{a} = \frac{1}{1} = a^{1-1} = a^{0} = 1$.

Corollary XXXIII.

155. The Product or Quotient of two Homologous Powers is equal to the Homologous Power raised from the Product or Quotient of their Roots; *i. e.* $a^m \times b^m = \overline{ab} \odot^m = \overline{ab}^m$. so $\frac{a^m}{b^m} = \frac{a}{b} \odot^m = \left|\overline{\frac{a}{b}}\right.^m$.

Corollary XXXIV.

156. The Product or Quotient made by two Homologous Powers has the same Exponent with those Powers; *i. e.* $a^m \times b^m = \left|\overline{ab}\right.^m$.

Corollary XXXV.

157. Fractions are involved to any Power by involving the Numerator and Denominator separately. Thus $\frac{a}{b} \odot^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ or $\left|\overline{\frac{a}{b}}\right.^3$ (In. 113.) so $\frac{a}{b} \odot^m = \frac{a^m}{b^m}$ or $\left|\overline{\frac{a}{b}}\right.^m$. Consequently

Corollary XXXVI.

158. The Root of a Fractional Power $\frac{a^m}{b^m}$ is had by extracting the Root of the Numerator, and Denominator separately; *i. e.* $\frac{a^m}{b^m} \omega^m = \frac{a}{b}$.

Corrollary XXXVII.

159. All Irrational or Surd Roots are incommensurate, but such as are extracted from commensurate Quantities are said to be *Commensurate in Power*.

Coollary XXXVIII.

160. Irrational Roots, tho' they cannot be had exactly in Numbers (In. 138.) yet they may be infinitely approached to, as lying betwixt the Roots of the next greater and next lesser Homologous Powers, *Ex. gr.* If it be required to extract the Square Root from the Number 12. Here because the next greater Square above 12 is 16, and the next less is 9, therefore the Square Root of 12, *i. e.* $12^{\frac{1}{2}}$ is some supposed Number between $9^{\frac{1}{2}}$ and $16^{\frac{1}{2}}$, *i. e.* between

 3 and

3 and 4: Which will be approached yet nearer by multiplying 12 × 100; for the next greater Square above 1200 is 1225 whose Root is 35, and the next lesser is 1156 whose Root is 34: Therefore $12^{\frac{1}{2}}$ is some supposed Number between $3\frac{4}{10}$ and $3\frac{5}{10}$. And this again will be approached nearer by multiplying 1200 by 100; for the next greater Square above 120000 is 120409 whose Root is 347, and the next lesser is 119716 whose Root is 346: Therefore $12^{\frac{1}{2}}$ is some supposed Number between $3\frac{46}{100}$ and $3\frac{47}{100}$. And thus by continuing to multiply the given Resolvend by 100, the Square Root will be approached nearer and nearer *ad infinitum*. The same way the Roots may be extracted from the other Irrational Powers, observing to multiply the Cube Resolvend by 1000, the Biquadrate by 10000, the first Sursolid by 100000, &c. For the easons of which, see (In. 185, 286, 288, Part 2.)

Corollary XXXIX.

161. The Difference between every irrational Root, and the next greater or next less rational one is less than Unity.

Theorem III.

162. If an Integer can have no rational Root in Integers, it can have none in Fractions.

Demonstration.

Every rational Root is an Aliquot Part of its Power (In. 127.) But that Part of any Integer which measures not the Whole is no Aliquot Part of the Whole (In. 37.) Therefore no rational Root to it. Q. E. D.

Theorem IV.

163. If the Quotient made by dividing any Power b^m by another Homologous to it c^m, be an Integer z, then the Root b divided by the Root c will also make an Integer in the Quotient.

Demonstration.

$\frac{b^m}{c^m} = z$ by Supposition $\therefore \frac{b}{c} = z^{\frac{1}{m}}$ (In. 146 and 158.) *i. e.* the Quotient made by dividing b by c is the m Root of the Integer z. But if, &c. (In. 162.) $\therefore \frac{b}{c}$ must be an Integer. Q. E. D.

CHAP. VIII.

Of Arithmetical and Geometrical Proportion.

PARTITION XV.

164. RATIO'S, as well as the Kinds of *Proportion* depending thereon, are two-fold, *viz. Arithmetical* and *Geometrical.*

DEFINITION LVII.

165. An *Arithmetical Ratio* is in respect of the Difference between the Antecedent and its Consequent, which is denoted with the Sign + before it in a *Ratio of greater Inequality*, and with the Sign — before it in a *Ratio of lesser Inequality. Ex. gr.* the *Arithmetical Ratio* of 8 to 2 is + 6, of 2 to 8 is — 6.

HYPOTHESIS XIV.

166. The Sign of the *Arithmetical Ratio* between two Terms is (··). *Ex. gr.* the Ratio of 8 to 2, is signified by 8 ·· 2, of 2 to 8 by 2 ·· 8.

DEFINITION LVIII.

167. *Arithmetical Proportion* is when two *Arithmetical Ratio's* have the same *common Difference*, as in these $9 \cdot\cdot 6 = 7 \cdot\cdot 4$, *i. e.* $9 - 6 = 7 - 4$. So $6 \cdot\cdot 9 = 4 \cdot\cdot 7$, *i. e.* $6 - 9 = 4 - 7$.

HYPOTHESIS XV.

168. Every *Arithmetical Proportion*, putting p indefinitely for the Antecedent of the former Ratio, q for the Antecedent of the latter, and d for the Common Difference, will, when both the Ratio's are of greater Inequality be represented by $p \cdot\cdot \overline{p-d} = q \cdot\cdot \overline{q-d}$; when both are of lesser Inequality, by $p \cdot\cdot \overline{p+d} = q \cdot\cdot \overline{q+d}$.

DEFINITION XV.

169. In every *Proportion* the Antecedent of the former Ratio and Consequent of the latter are called *Extreams*, and the other two Terms *Means.*

PARTITION XVI.

170. *Arithmetical Proportion* is either *Discontinued* or *Continued.*

DEFINITION LX.

171. *Discontinued Arithmetical Proportion* is when the *Mean Terms* taken as Antecedent and Consequent have a different Ratio to one another from that of the other Antecedents to their Consequents; as in the Examples (In. 167.)

DEFINITION LXI.

DEFINITION LXI.

172. *Arithmetical Proportion Continued* is when the *Mean Terms* taken as Antecedent and Consequent have the same Ratio to one another with that of the other Antecedents to their Consequents; or when the Ratio of the second Term to the third is the same with that of the first to the second, or of the third to the fourth: As in these $12 \cdot\cdot 9 = 6 \cdot\cdot 3$, or $3 \cdot\cdot 6 = 9 \cdot\cdot 12$. where $12 \cdot\cdot 9 = 9 \cdot\cdot 6 = 6 \cdot\cdot 3$, or $3 \cdot\cdot 6 = 6 \cdot\cdot 9 = 9 \cdot\cdot 12$.

DEFINITION LXII.

173. *Extream* and *Mean Proportion Arithmetical* is when a Number is divided into two such Parts that the lesser Part has the same *Arithmetical Ratio* to the greater that the greater has to the Whole; as the Number 9 divided into 6 and 3, because $9 \cdot\cdot 6 = 6 \cdot\cdot 3$, or $3 \cdot\cdot 6 = 6 \cdot\cdot 9$.

THEOREM V.

174. In every *Arithmetical Proportion* the Sum of the two *Extreams* is equal to the Sum of the two *Means*.

Demonstration.

Any Arithmetical Proportion, if *Ex. gr.* the Ratio's be of greater Inequality, will be expressed by $p \cdot\cdot \overline{p-d} = q \cdot\cdot \overline{q-d}$ (In. 168) But $p + \overline{q-d} = \overline{p-d} + q$ by Inspection; Therefore, &c. Q. E. D.

COROLLARY XL.

175. If the Sum of any two Terms equals the Sum of any other two, then are these four Terms in Arithmetical Proportion; *i. e.* If *Ex. gr.* $b + f = c + d$ then $b \cdot\cdot c = d \cdot\cdot f$.

COROLLARY XLI.

176. If the Sum of any two Terms be double a third Term, then are those three Terms in Arithmetical Proportion; *i. e.* if $b + d = 2c$, then $b \cdot\cdot c = c \cdot\cdot d$.

PROBLEM XI.

177. From any three Terms b, c, d given, to find a Fourth Term in Arithmetical Proportion.

Effection.

Add the second and third Terms together, and from their Sum $c + d$ subtract the first Term b, the Remainder $\overline{c + d - b}$ will be the fourth Proportional sought; *i. e.* $b \cdot\cdot c = d \cdot\cdot \overline{c+d-b}$. Q. E. E.

PROBLEM XII.

PROBLEM XII.

178. Between any two Terms b and d given to find a Mean Arithmetical Proportional.

Effection.

Add the two given Terms together, and take the half of their Sum and the Quotient $\frac{b+d}{2}$ will be the Mean Proportional required; *i. e.* $b \cdot\cdot \frac{b+d}{2} = \frac{b+d}{2} \cdot\cdot d$. Q. E. E.

PROBLEM XIII.

179. To divide any given Number b into Extream and Mean Proportion Arithmetical.

Effection.

Divide the given Number b into three equal Parts; then $\frac{1}{3}b \cdot\cdot \frac{2}{3}b = \frac{2}{3}b \cdot\cdot b$. Q. E. E.

DEFINITION LXIII.

180. A *Geometrical Ratio*, or *Ratio* properly so called, is in respect of the Quotient arising from the Division of the Antecedent by its Consequent.

DEFINITION LXIV.

181. And that Quotient is stiled the *Nominator* of the Ratio; because the Ratio is denominated by it.

DEFINITION LXV.

182. If the lesser Term be an Aliquot Part of the greater, the *Ratio of greater Inequality* is stiled *Multiple*, and the *Ratio of lesser Inequality Submultiple*. In particular, if the Nominator of the *Ratio* be 2, it is called *Double*, if $\frac{1}{2}$, *Subduple*; if 3, it is called *Triple*, if $\frac{1}{3}$, *Subtriple*; if 4, *Quadruple*, if $\frac{1}{4}$, *Subquadruple*, &c.

DEFINITION LXVI.

183. If the greater Term contain the lesser Term once, and an Aliquot Part of it self over; the *Ratio of greater Inequality* is called *Superparticular*, and the *Ratio of lesser Inequality Subsuperparticular*. Particularly if the Nominator of the *Ratio* be $1\frac{1}{2}$ or $\frac{3}{2}$ it is stiled *Sesquialter*, if $\frac{2}{3}$ *Subsesquialter*; if $1\frac{1}{3}$ or $\frac{4}{3}$ *Sesquitertian*; and if $\frac{3}{4}$ *Subsesquitertian*; if $1\frac{1}{4}$, or $\frac{5}{4}$ *Sesquiquartan*, and if $\frac{4}{5}$ *Subsesquiquartan*, &c.

HYPOTHESIS XVI.

184. The Sign of the *Geometrical Ratio* between two Terms is (:) Thus, *Ex. gr.* the *Quadruple Ratio* of 8 to 2 is signified by 8 : 2 or $\frac{8}{2}$; the *Subquadruple Ratio* of 8 to 2 is denoted by 2 : 8 or $\frac{2}{8} = \frac{1}{4}$ (In 87.)

PARTITION XVII.

185. *Geometrical Proportion* is divided into *Simple*, *Multiplicate*, *Harmonical*, and *Contraharmonical*.

DEFINITION LXVII.

186. *Simple Geometrical Proportion* is when two *Geometrical Ratio's* have the ſame *Common Nominator* of the *Ratio*; as in theſe $12 : 6 = 8 : 4$, and $6 : 12 = 4 : 8$, which are thus read: As 12 is to 6, ſo is 8 to 4; and as 6 is to 12, ſo is 4 to 8.

PARTITION XVIII.

187. *Simple Geometrical Proportion* is either *Diſcontinued*, or *Continued*, after the ſame manner with *Arithmetical Proportion* (In. 171 and 172.)

HYPOTHESIS XVII.

188. Every *Simple Geometrical Proportion* putting p indefinitely for the Antecedent of the former Ratio, q for the Antecedent of the latter, and r for the Nominator of the Ratio, will, when the *Ratio's* are of *greater Inequality*, be repreſented by $p : \frac{p}{r} = q : \frac{q}{r}$, when *of leſſer Inequality*, by $p : pr = q : qr$. Or putting $z = \frac{1}{r}$ in the former Caſe, and $z = \frac{r}{1} = r$ in the latter, then every *Geometrical Proportion* will be expreſſed univerſally; thus, $p : zp = q : zq$, or $\frac{p}{zp} = \frac{q}{zq}$.

THEOREM VI.

189. In every *Simple Geometrical Proportion* the Product of the two *Extreams* is equal to the Product of the two *Means*.

Demonſtration.

Every Geometrical Proportion may be expreſſed by $p : zp = q : zq$ (In. 188.) But $p \times zq = zp \times q$ (In. 85.) Therefore, &c. Q. E. D.

COROLLARY XLII.

190. If the Product of any two Terms equals the Product of other two, then are thoſe Terms in Geometrical Proportion; *i. e.* if *Ex. gr.* $bf = cd$, then $b : c = d : f$, or $\frac{b}{c} = \frac{d}{f}$, or $\frac{c}{b} = \frac{f}{d}$.

COROLLARY XLIII.

Corollary XLIII.

191. If the Product of any two Terms equals the Square of a third Term, then are those three Terms in Geometrical Proportion; *i. e.* if $bd = cc$, then $b : c = c : d$, or $\frac{b}{c} = \frac{c}{d}$, or $\frac{c}{b} = \frac{d}{c}$.

Corollary XLIV.

192. If any four Terms are in Geometrical Proportion, *viz.* $p : pz = q : qz$, then by (In. 188.) they will appear to be so.

1	Directly	$p : zp = q : zq$
2	Invertedly	$zp : p = zq : q$
3	Alternately	$p : q = zp : zq$
4	Compoundedly	$p + zp : zp = q + zq : zq$
5	Conversely	$p + zp : p = q + zq : q$
6	Dividedly	$p - zp : zp = q - zq : zq$
7		$p - zp : p = q - zq : q$
8	By a Syllepsis	$p : zp = p + q : zp + zq$
9	By a Dialepsis	$p : zp = p - q : zp - zq$

Corollary XLV.

193. If one Rank or Series of Terms in Geometrical Proportion be orderly multiplied or divided by another, the respective Products or Quotients will be also in Geometrical Proportion; *i. e.*

$$\begin{array}{rl} \text{If} & p : zp = q : zq \\ \text{And} & a : ea = y : ey \\ \text{Then} & pa : zpea = qy : zqey \\ \text{And} & \frac{p}{a} : \frac{zp}{ea} = \frac{q}{y} : \frac{zq}{ey} \text{ (In. 188.)} \end{array}$$

Corollary XLVI.

194. Whence, if Quantities are in Geometrical Proportion to one another, their Homologous Powers, and Homologous Roots are so also; *i. e.* if $a : b = c : d$, then $a^m : b^m = c^m : d^m$, and $a^{\frac{1}{m}} : b^{\frac{1}{m}} = c^{\frac{1}{m}} : d^{\frac{1}{m}}$.

Corollary XLVII.

195. In Multiplication the Ratio of Unity to one Factor is the same with the Ratio of the other Factor to the Product: Or, as Unity is to the Multiplier, so is the Multiplicand to the Product; and consequently in Division, As the Divisor is to the Dividend, so is Unity to the Quotient, &c.

Problem XIV.

Problem XIV.

196. From any three given Terms b, c, d to find a fourth a in Geometrical Proportion.

Effection.

Multiply the second and third Terms together, and divide the Product cd by the first Term b; the Quotient $\frac{cd}{b}$ will be the fourth Proportional sought; *i. e.* $b : c = d : \frac{cd}{b} = a$. Q. E. E.

Scholium III.

197. The Effection of this Problem is commonly called the *Golden Rule*, by reason of its exceeding Usefulness in practical Arithmetick: Or otherwise, it is termed the *Rule of Three*, because here are always three Terms given to find a fourth: And it is either *Direct*, when the Terms stand in their natural Order, as above; or *Inverse*, when they stand in an Order inverted; *Ex. gr.* If it be demanded, what will be the Price of 9 Yards of that Piece of Linnen, three Yards of which are sold for 12 *Shillings*. Here putting a for the Price of the 9 Yards sought, the Terms will stand thus, $\overset{Y.}{3} : \overset{Y.}{9} = \overset{S.}{12} : \overset{S.}{a}$; *i. e.* as 3 Yards are to 9 Yards, so are 12 *Shillings* to a *Shillings*; or alternately thus, $\overset{Y.}{3} : \overset{S.}{12} = \overset{Y.}{9} : \overset{S.}{a}$; *i. e.* as three Yards are to 12 *Shillings*, so are $\overset{S.}{9}$ Yards to a *Shillings*; by both which Proportions $a = \frac{12 \times 9}{3} = \frac{108}{3} = 36$ according to the Effection above.

But if it be demanded what Length of Ground of 9 Perches wide must be given in Exchange for an Enclosure of 35 Perches long and 21 broad, Quantity for Quantity. Here putting a for the Length of Ground demanded of 9 Perches wide, then because the Product of $9 \times a$ is to be the same with the Product of 21×35 by the Question: Hence have we this Proportion $\overset{P.}{9} : \overset{P.}{21} = \overset{P.}{35} : \overset{P.}{a}$ (In. 190.) *i. e.* As 9 Perches wide is to 21 Perches wide, so is 35 Perches long to a Perches long; Whence $a = \frac{21 \times 35}{9} = \frac{735}{9} = 81\frac{6}{9}$. And this is what is called the *Rule of Three Inverse*, of which take another Example as follows;

Let it be required to determine how many Hours 9 Cocks will take up in emptying a Cistern of Water, which is emptied by 12 Cocks of the same Wideness in 6 Hours. Here putting a for the Hours required, then because the Product of $9 \times a$ is to be the same with the Product of 6×12 by the Question,

Queſtion; Hence have we this Proportion $9 : 12 = 6 : a$. Therefore $a = \frac{12 \times 6}{9} = \frac{72}{9} = 8$ Hours.

Scholium IV.

198. It ſometimes happens that five Terms are given in Geometrical Proportion to find a ſixth, which is called the *Double* or *Compound Rule* of *Three*: And this again is either *Direct*, when the Queſtion is reſolved by two *Direct Proportions*; or *Inverſe*, when it is reſolved by one *Direct*, and another *Inverſe Proportion*. An Example of the former Sort is the following one.

If 100 or b Pounds Sterling in 12 or p Months gain 4 or d Pounds Intereſt, how much will 650 or c Pounds gain in 18 or q Months? Here becauſe the Queſtion lies upon the gain of 650 Pounds in $q = 18$ Months, therefore theſe two Terms muſt be the Conſequents of the firſt Ratio in either Proportion; and $d = 4$ Pounds is the Antecedent of the latter Ratio in the firſt Proportion.

Thus $b : c = d : \frac{cd}{b}$, *i.e.* $\overset{L.}{100} : \overset{L.}{650} = \overset{L.}{4} : \frac{4 \times 650}{100} = \overset{L.}{26}$, the Intereſt of $\overset{L.}{650}$ for 12 Months, to be the Antecedent of the latter Ratio in the ſecond Proportion.

$p : q = \frac{dc}{b} : \frac{dcq}{bp}$ i. e. $\overset{M.}{12} : \overset{M.}{18} = \overset{L.}{26} : \frac{26 \times 18}{12} = \overset{L.}{39}$ the Anſwer required. Otherwiſe

$p : q = d : \frac{dq}{p}$ i. e. $\overset{M.}{12} : \overset{M.}{18} = \overset{L.}{4} : \frac{4 \times 18}{12} = \overset{L.}{6}$ the Intereſt of 100 for 18 Months, to be the Antecedent of the latter Ratio of the ſecond Proportion.

$b : c = \frac{dq}{p} : \frac{dcq}{bp}$ *i. e.* $\overset{L.}{100} : \overset{L.}{650} = 6 : \frac{6 \times 650}{100} = 39$ as before.

Whence it is plain that the Queſtion may be performed by Compoſition at one Operation; as follows,

$bp : cq = d : \frac{dcq}{bp}$ i. e. $\overset{L.}{100} \times \overset{M.}{12} : \overset{L.}{650} \times \overset{M.}{18} = \overset{L.}{4} : \overset{L.}{39}$. Therefore the Theorem for all Queſtions of this Kind is $\frac{dcq}{bp}$, as is further illuſtrated (In. 341.)

The *Compound Rule of Three Inverſe* is performed as in the following Example.

K It

It is required to find what Principal $= a$ well gain $z = 39$ Pounds Sterling in $q = 18$ Months, at $d = 4$ Pounds, *per Cent.* $= 100$ Pounds $= b$, *per Annum* $= 12$ Months $= p$. The first Operation is in direct Proportion (In. 196.) thus,

$$\overset{M.}{p} : \overset{M.}{q} = \overset{L.}{d} : \overset{L.}{\frac{dq}{p}}$$ i. e. $12 : 18 = 4 : \overset{L.}{6}$, the Interest of $b = 100$ Principal for 18 Months. Then since the Product of 100×39 must equal the Product of $a \times 6$, or $b \times z = a \times \frac{dq}{p}$ by the Question; therefore

$\frac{dq}{p} : z = b : a$ (In. 190.) *i. e.* $6 : 39 = 100 : \overset{L.}{650}$ or $\overset{L.}{a}$ Pounds, the Principal required; whence representing things duly as above, the Theorem for all Questions of this Kind is $\frac{bzp}{dq} = a$.

Scholium V.

109. When the Antecedent and Consequent of a given Ratio are either, or each of them, a Compound of more Terms than one, they are to be reduced (In. 43 and 341.) to single Terms of the same Denomination, as in this Question;

If $\overset{C.}{15}$ $\overset{Qrs.}{1}$ $\overset{lb.}{20}$ (*i. e.* 15 Hundred Weight, 1 Quarter, and 20 Pound *Weight of any thing cost* $\overset{L.}{40}$ $\overset{S.}{16}$ (*or* 40 Pound 16 Shillings Sterling) *what will* $\overset{C.}{1}$ $\overset{qr.}{1}$ $\overset{lb.}{4}$ *cost at that rate?* Which reduced makes this Proportion; $\overset{lb.}{1728} : \overset{lb.}{144} = \overset{S.}{816} : \frac{144 \times 816}{1728} = \overset{S.}{68} = \overset{L.}{3}\ \overset{S.}{8}$, the Answer required.

Problem XV.

200. Between two Terms b, c, to find a Mean Geometrical Proportional a.

Effection.

Multiply the two given Numbers together, and the Square Root of the Product $\overline{bc}^{\frac{1}{2}}$ will be the mean Proportional sought $= a$, i. e. $b : a = a : c$, or $b : \overline{bc}^{\frac{1}{2}} = \overline{bc}^{\frac{1}{2}} : c$ (In. 189.) Q. E. E. *Ex. gr.* If $b = 4$ and $c = 9$ then $a = \overline{4 \times 9}^{\frac{1}{2}} = 36^{\frac{1}{2}} = 6$.

Definition LXVIII.

201. *Multiplicate Proportion* is that of the *Ratio* of two *Homologous Powers* in respect of the Ratio of their *Roots*. *Submultiplicate Proportion* is that of the *Ratio* of two *Roots* in respect of their *Homologous Powers*. In particular, the

the *Ratio* of two *Squares* is said to be *Duplicate*; of two *Cubes*, *Triplicate*; of two *Biquadrates*, *Quadruplicate*, &c. of that of their *Roots*: and *vice versâ*, the *Ratio* of two *Square Roots* is said to be *Subduplicate*, of two *Cube Roots*, *Subtriplicate*; of two *Biquadrate Roots*, *Subquadruplicate*, &c. in respect of the *Ratio* of their *Powers*. And the same Denomination is given to the *Ratio* between any two Products of the same Number of Dimensions in respect of the *Ratio* between their *Homologous Factors*. *Ex. gr.* the Ratio of 300×10 to 600×20, or of 3000 to 12000, is in *Duplicate Proportion* to the *Ratio* of 300 to 600, or of 10 to 20; and the contrary *Subduplicate*. The *Ratio* of $2 \times 3 \times 4$ to $6 \times 9 \times 12$ is *Triplicate* in respect of the Ratio of 2 to 6, of 3 to 9, or of 4 to 12, and contrarily *Subtriplicate*, &c.

Definition LXIX.

202. *Harmonical*, or *Musical Proportion* is when of three Terms, a, b, c, (Ex. gr. $a = 6$, $b = 8$, $c = 12$) there is the same *Geometrical Ratio* between the Difference of the first and second, and the Difference of the second and third, as there is between the first and third; i. e. $b - a : c - b = a : c$. Or when of four Terms a, b, c, d (Ex. gr. $a = 6$, $b = 8$, $c = 12$, $d = 18$) there is the same *Geometrical Ratio* between the Difference of the first and second, and the Difference of the third and fourth, as there is between the first and fourth; i. e. $b - a : d - c = a : d$.

Definition LXX.

203. *Contraharmonical Proportion* is when of three Terms a, b, c, (*Ex gr.* $a = 3$, $b = 5$, $c = 6$) the Difference between the first and the second $(b - a)$ is to the Difference between the second and the third $(c - b)$ as the third is to the first; *i. e.* $b - a : c - b = c : a$.

Chap. IX.

Of Arithmetical and Geometrical Progression.

Definition LXXI.

204. A *RITHMETICAL Progression* is when a Series or Rank of Homogeneous Terms do encrease or decrease by the same *Arithmetical Ratio*; as

1, 2, 3, 4, 5, 6, 7, 8, 9, *&c.* } whose *Common Difference* is 1,
9, 8, 7, 6, 5, 4, 3, 2, 1 } therefore stiled Laterals.

2, 4, 6, 8, 10, 12, 14, 16, 18, *&c.* } whose *Common Difference* is 2.
18, 16, 14, 12, 10, 8, 6, 4, 2 }

And so for any other Rank whose *Common Difference* is 3, 4, 5, 6, *&c.*

Hypothesis XVIII.

Hypothesis XVIII.

205. The Sign of Arithmetical Progreſſion is $\div$

Hypothesis XIX.

206. If the Species l be put indefinitely for the leaſt Term of any Series in $\div$, g for the greateſt, d for the common Difference, n for the Number of Terms, and s for the Sum of the whole Series; then every encreaſing Series will be thus repreſented l, $l+d$, $l+2d$, $l+3d$, $l+4d$, $l+5d$, &c. according to n the Number of Terms: And every decreaſing Series, thus, g, $g-d$, $g-2d$, $g-3d$, $g-4d$, $g-5d$, &c. till we come to l the leaſt Term.

Corollary XLVIII.

207. The greateſt Term of any Series in $\div$ is equal to the leaſt Term added to as many times the common Difference, as there are Terms in the Series leſs one, i. e. $g = l + d \times \overline{n-1}$ or $g = l + nd - d$.

Corollary XLIX.

208. If $l = o$, then the encreaſing Series in $\div$ will be thus, o, d, $2d$, $3d$, $4d$, $5d$, &c. till we come to $g = nd - d$: The decreaſing Series thus, $nd - d$, $nd - 2d$, $nd - 3d$, $nd - 4d$, $nd - 5d$, &c. till we come to $l = nd - nd = o$.

Corollary L.

209. In any Series of Terms in $\div$ the Difference between all ſuch Terms as are equally diſtant from one another is equal, as conſiſting of the *Common Difference* the ſame Number of times repeated; *Ex. gr.* the Difference between the firſt and third Terms is the ſame with the Difference between the ſecond and fourth, and between the ſeventh and ninth, *&c.* ſo alſo the Difference between the ſecond and ſixth Terms, is the ſame with the Difference between the fourth and eighth, and between the fifth and Ninth, *&c.* Therefore

Corrollary LI.

210. If any Number of Terms be in $\div$, the leaſt Term the greateſt Term, and any two middle Terms which are equally diſtant from the leaſt and greateſt, will be four Terms in Arithmetical Proportion (In. 167.) and conſequently the Sum of the two Extreams will equal the Sum of any two equidiſtant Means. (In. 174.)

Corollary LII.

211. If any odd Number of Terms be in $\div$ then the leaſt Term, the greateſt Term, and the middle Term will be three Terms in Arithmetical

tical Proportion (In. 167), and consequently the Sum of the two Extreams will be double the Mean (In. 176.)

Theorem VII.

212. In any Series of Terms in $\div$, if the Sum of the greatest and least Terms be multiplied into One half the Number of Terms, the Product will be twice the Sum of the whole Series; i. e. $\frac{n}{2} \times \overline{l+g} = s$.

Demonstration.

$l+g$ is equal to the Sum of every two equidistant Means (In 209.) and to Double the middle Term of all, if the Number of Terms be odd. (In 210.) But if every two Terms in the whole Rank be thus summ'd up, there can be but half so many Sums, or $\overset{S}{\overline{l+g}}$, as the Number of Terms n, $\therefore$ the Sum of all the Terms in the Series must equal $\frac{n}{2}$ times $\overline{l+g}$: i. e. $s = \frac{n}{2} \times \overline{l+g}$ or $\frac{nl+ng}{2}$. Q. E. D.

Corollary LIII.

213. Therefore the Sum of every Rank of Laterals beginning with Unity (*i e.* when $l = d = 1$, consequently $g = n$) will be $\frac{nn+n}{2}$.

Corollary LIV.

214. Or if $l = o$, then $s = \frac{n}{2} \times \overline{o+g} = \frac{ng}{2}$.

Definition LXXII.

215. *Geometrical Progression* is when a Series or Rank of Homogeneous Terms do encrease or decrease by the same Geometrical Ratio, as,

1, 2, 4, 8, 16, 32, 64, 128, 256, &c.
256, 128, 64, 32, 16, 8, 4, 2, 1. } whose Nominator is 2.

1, 3, 9, 27, 81, 243, 729, 2187, 6561, &c.
6561, 2187, 729, 243, 81, 27, 9, 3, 1 } whose Nominator is 3.

And so for any other Rank where the Nominator is 4, 5, 6, 7, &c.

Scholium VI.

216. Observe that the Nominator of the Ratio here always belongs to the Ratio of lesser Inequality.

Hypothesis XX.

217. The Sign of Geometrical Progression is $\because$

Hypothesis XXI.

218. If the Species l be put indefinitely for the least Term of any Series in $\div\div$, g for the greatest Term, r for the Nominator of the Ratio, n for the Number of Terms, s for the Sum of the whole Series: Then every encreasing Series will be thus represented, l, lr, lr^2, lr^3, lr^4, lr^5, &c. And every decreasing Series thus, g, $\frac{g}{r^1}$, $\frac{g}{r^2}$, $\frac{g}{r^3}$, $\frac{g}{r^4}$, $\frac{g}{r^5}$, &c. according to n the Number of Terms. Whence

Corollary LV.

219. The greatest Term of any Series in $\div\div$ is equal to the least Term multiplied into that Power of the Nominator, whose Exponent is the Number of Terms less one; i. e. $g = lr^{n}$

Corollary LVI.

220. If $l = 1$ then will the encreasing Series in $\div\div$ be thus represented, 1, r, r^2, r^3, r^4, r^5, r^6, &c. till we come to $g = r^{n-1}$. And if $g = 1$ the decreasing Series will be 1, $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, $\frac{1}{r^4}$, $\frac{1}{r^5}$, &c. or which is the same thing r^0, r^{-1}, r^{-2}, r^{-3}, r^{-4}, r^{-5}, &c. (In. 154.)

Corollary LVII.

221. Therefore every Scale of Powers is a Rank of Terms in Geometrical Progression, whose first Term is Unity, and Nominator is the Root. And the Exponents of the Scale are a Rank of Terms in Arithmetical Progression whose first Term is o, and Common Difference is Unity.

Corollary LVIII.

222. In every Series of Terms in $\div\div$, the Ratio between all such Terms as are equally distant from one another is the same.

Corollary LIX.

223. In any Number of Terms in $\div\div$, the two Extreams and any two Means which are equally distant from those Extreams are four Terms in Geometrical Proportion (In. 186.) consequently the Product of the two Extreams is equal to the Product of any two equidistant Means (In. 189.)

Corollary LX.

224. If any odd Number of Terms be in $\div\div$, the last Term, the greatest Term, and the middle Term are three Terms in Geometrical Proportion (In. 186.)

(In. 186.) and consequently the Product of the two Extreams is equal to the Square of the Mean.

Theorem VIII.

225. In any Series of Terms in $\div\!\!\div$, the Sum of all the Terms, except the greatest, multiplied into the Nominator of the Ratio will be equal to the Sum of all the Terms except the least; i. e. $\overline{s-g} \times r = s - l$, or $sr - gr = s - l$.

Demonstration.

Suppose any Series of Terms in $\div\!\!\div$ whose least Term is l, Nominator r, and last Term g; Ex. gr. lr^5, i. e. $lr^5 = g$ (In. 218.) Then $s - g = l + lr + lr^2 + lr^3 + lr^4$, and $s - l = lr + lr^2 + lr^3 + lr^4 + lr^5 = r \times \overline{l + lr + lr^2 + lr^3 + lr^4} = r \times \overline{s-g}$ (In. 71). But this will always be, let the Terms be many or few (In. 214.) $\therefore$ $s - l = r \times \overline{s-g}$ (In. 21.) Q. E. D.

Chap. X.

Of Polygonal Numbers.

Definition LXXIII.

226. POLYGONAL *Numbers* are such as arise from the Addition of a Series of Numbers in $\div$ beginning with Unity; and are stiled,

First, *Triangulars*, or *Trigons*, when the common Difference is 1.

Arith. Progr. { 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c.*
Trigons { 1, 3, 6, 10, 15, 21, 28, 36, 45, *&c.*

Secondly, *Quadrangulars* or *Tetragons*, when the common Difference is 2.

Arith. Progr. { 1, 3, 5, 7, 9, 11, 13, 15, 17, *&c.*
Tetragons. { 1, 4, 9, 16, 25, 36, 49, 64, 81, *&c.*

Thirdly, *Pentagons*, when the Common Difference is 3.

Arith. Progr. { 1, 4, 7, 10, 13, 16, 19, 22, 25, *&c.*
Pentagons. { 1, 5, 12, 22, 35, 51, 70, 92, 117, *&c.*

Fourthly, *Hexagons*, when the common Difference is 4.

Arith. Progr. { 1, 5, 9, 13, 17, 21, 25, 29, 33, *&c.*
Hexagons. { 1, 6, 15, 28, 45, 66, 91, 120, 153, *&c.*
&c. *&c.* *&c.*

Definition LXXIV.

227. The *Side* or *Root* of a *Polygon* is the Number of Terms in the $\div$, which are sum'd up for forming it.

Definition LXXV.

DEFINITION LXXV.

228. The *Denominator* of the *Polygon*, which otherwise is called the *Number of its Angles*, is the first *Polygonal Number* of its Kind, next after Unity, from whence each Kind receives its Name; as the Number 3 in a *Trigon*, 4 in a *Tetragon*, 5 in a *Pentagon*, 6 in a *Hexagon*, 7 in a *Heptagon*, 8 in an *Octagon*, &c.

HYPOTHESIS XXII.

229. In the Arithmetick of *Polygons* put n for the *Side* or *Root* of the *Polygon*, or the Number of Terms in ∸ from whence it is formed, d for the Common Difference of the *Arithmetical Series*, g or G for the greatest Term in the said ∸, S or P for the *Polygon* it self, and D for its Denominator, or the Number of its Angles.

COROLLARY LXI.

230. The Common Difference of the ∸ is always equal to the Denominator of the Polygon formed from it, less 2; i. e. $d = D - 2$.

COROLLARY LXII.

231. It appears also that every Scale of *Polygons* (*i. e.* the *Trigon*, *Tetragon*, *Pentagon*, &c.) formed from the same *Root* n is a Rank of Terms in ∸ whose least Term is n, and whose Common Difference is that *Trigon* which has for its *Root* $n - 1$; *Ex. gr.* let $n = 4$, then because the *Trigon* is 6 whose *Root* is $n - 1 = 3$; therefore the *Trigon*, whose *Root* is $n = 4$, is $n + 6 = 10$; the *Tetragon* is $10 + 6 = 16$; the *Pentagon* is $16 + 6 = 22$; the *Hexagon* is $22 + 6 = 28$; the *Heptagon* is $28 + 6 = 34$, *&c.* And the same for any other *Root* or *Side*.

DEFINITION LXXVI.

232. *Pyramidal Numbers* are such as are formed by the Addition of a Series of *Polygons* after the same manner as that Series of *Polygons* were formed by the Addition of a Series of Terms in ∸. And the Sums of those *first Pyramidals* the same way collected are called *Second Pyramidals*; the Sums of those *Second Pyramidals*, third Pyramidals, *&c. ad infinitum*; which in particular are stiled *Triangular*, *Quadrangular*, *Pentagonal*, &c. according as they are formed from a Series of *Polygons* that are *Trigons*, *Tetragons*, *Pentagons*, &c. *Ex. gr.* The Genesis of *Pyramidals Triangular*, are as follows:

Units

Unites	1	1	1	1	1	1	1	1	1	&c.
Laterals	1	2	3	4	5	6	7	8	9	&c.
Trigons	1	3	6	10	15	21	28	36	45	&c.
1st *Pyramidals.*	1	4	10	20	35	56	84	120	165	&c.
2d *Pyramidals.*	1	5	15	35	70	126	210	330	495	&c.
3d *Pyramidals.*	1	6	21	56	126	252	462	792	1287	&c.
4th *Pyramidals.*	1	7	28	84	210	462	924	1716	3003	&c.
5th *Pyramidals.*	1	8	36	120	330	792	1716	3432	6435	&c.
6th *Pyramidals.*	1	9	45	165	495	1287	3003	6435	12870	&c.
&c.	&c.								&c.	

From whence it will be easy to conceive the Formation of *Pyramidals Quadrangular*, *Pentagonal*, &c.

CHAP. XI.

Of Combinations and Permutations.

DEFINITION LXXVII.

233. COMBINATION is the taking together of all the different *Two's*, or *Pairs*; the different *Three's*, or *Ternaries*; the different *Four's*, or *Quaternaries*; *Fives*, or *Quinaries*, &c. that any given Number of Things can admit of.

PARTITION XIX.

234. *Combinations* are either *Simple* or *Redundant*.

DEFINITION LXXVIII.

235. A *Simple Combination* is when the same thing in each *Combination* occurs but once.

THEOREM IX.

236. The *Simple Combinations* of things by *Pairs* proceed in a Rank of *Trigons*, whose Root is the Number of things to be combined less one, or $n-1$; by *Ternaries* in a Rank of *first Pyramidals Triangular*, whose Root is $n-2$; by *Quaternaries* in a Rank of *second Pyramidals Triangular*, whose

 Root

Root is $n - 3$; by *Quinaries*, in a Rank of *third Pyramidals*, whose Root is $n - 4$, &c. *ad infinitum*.

Demonstration.

Con 1. If $n = 2 = A, B$, then the Number of *Pairs* in the Things to be *combined* is 1 (a *Trigon*, whose Root is $n - 1 = 1$, In. 226) viz. *AB*: If $n = 3 = A, B, C$, then the Number of *Pairs* are 3 (a *Trigon*, whose Root is $n - 1 = 2$, In. 226) viz. *AB*, *AC*, *BC*: If $n = 4 = A, B, C, D$, then the Number of *Pairs* are 6 (a *Trigon*, whose Root is $n - 1 = 3$) viz. *AB*, *AC*, *AD*, *BC*, *BD*, *CD*: If $n = 5 = A, B, C, D, E$, the Number of *Pairs* are 10 (a *Trigon*, whose Root is $n - 1 = 4$) viz. *AB*, *AC*, *AD*, *AE*, *BC*, *BD*, *BE*, *CD*, *CE*, *DE*. But this Law will still hold, if $n = 6$, $n = 7$, $n = 8$, &c. *ad Infinitum*. ∴ Therefore the Number of *Pairs*, in any Number of Things n, is a *Trigon*, whose Root or Side is $n - 1$.

Con. 2. If $n = 3 = A, B, C$, the Number of *Ternaries* is 1 (a *first Pyramidal Triangular*, whose *Root* is $n - 2 = 1$, In 232) viz. *ABC*: If $n = 4 = A, B, C, D$, then the Number of *Ternaries* are 4 (a *first Pyramidal Triangular*, whose Root is $n - 2 = 2$, In. 231) viz. *ABC*, *ABD*, *ACD*, *BCD*: If $n = 5 = A, B, C, D, E$, the Number of *Ternaries* are 10 (a *first Pyramidal*, whose Root is $n - 2 = 3$, In. 232) viz. *ABC*, *ABD*, *ABE*, *ACD*, *ACE*, *ADE*, *BCD*, *BDE*. But this Law will still hold, if $n = 6$, $n = 7$, $n = 8$, $n = 9$, &c. ∴ The Number of *Ternaries* in any Number of Things n, is a *Pyranidal Triangular*, whose Root or Side is $n - 2$.

Con. 3. If $n = 4 = A, B, C, D$, then the Number of *Quaternaries* is 1 (a *second Pyramidal Triangular*, whose Root is $n - 3 = 1$, In. 232) viz. *ABCD*. If in $n = 5 = A, B, C, D, E$, the Number of *Quaternaries* are 5 (a *second Pyramidal*, whose Root is $n - 3 = 2$. In. 232) viz. *ABCD*, *ABCE*, *ABDE*, *ACDE*, *BCDE*: If $n = 6 = A, B, C, D, E, F$, the Number of *Quaternaries* are 15 (a *second Pyramidal*, whose Root is $n - 3 = 3$ In. 232) viz. *ABCD*, *ABCE*, *ABCF*, *ABDE*, *ABDF*, *ACDE*, *ACDF*, *ADEF*, *BCDE*, *BCDF*, *BDEF*, *BCEF*, *ABEF*, *ACEF*, *CDEF*: But this will still hold. if $n = 7$, $n = 8$, $n = 9$, $n = 10$, &c. ∴ the Number of *Quaternaries*, in any Number of Things n, is a *second Pyramidal Triangular*, whose Root or Side is $n - 3$.

Con. 4. And by the same Method of Induction, we shall find that the Number of all the Quinaries, that any Number of Things can admit of, will be a *third Pyramidal Triangular*, whose Root is $n - 4$; the Number of all the *Senaries*, a *fourth Pyramidal*, whose Root is $n - 5$; of all the *Septenaries*, a *fifth Pyramidal*, whose Root is $n - 6$, &c. Whence we may infer the Certainty of the Theorem. Q. E. D.

Definition LXXIX.

Definition LXXIX.

237. I call that a *Redundant Combination*, wherein the same thing occurs oftner than once, as in these, *AA*, *AAB*, *BBAC*, &c.

Theorem X.

238. The Number of all the *Possible Combinations (Simple* and *Redundant)* of any given Number of things n; by *Pairs* is a *Trigon*, by *Ternaries* a first Pyramidal Triangular, by *Quaternaries* a second Pyramidal Triangular, by *Quinaries* a third Pyramidal, by Senaries a fourth, *&c. ad Infinitum* whose Root is n.

Demonstration.

Con. 1. If $n = 1 = A$, the Number of all the *possible Pairs* is 1 (a *Trigon* whose Root is $n = 1$ In. 226) viz. *AA*: If $n = 2$, *A*, *B*, the Number of all the *possible Pairs* are three (a *Trigon*, whose Root is $n = 2$) viz. *AA*, *AB*, *BB*: If $n = 3$, *A*, *B*, *C*, the Number of all the *possible Pairs* are 6 (a *Trigon*, whose Root is $n = 3$) viz. *AB*, *AA*, *AC*, *BB*, *BC*, *CC*. But this will still hold, if $n = 4$, $n = 5$, $n = 6$, &c. ∴ the Number of all the *Possible Pairs*, in any given Number of things to be combined n, is a *Trigon*, whose Root is n.

Con. 2. If $n = 1 = A$, the Number of all the *possible Ternaries* is 1 (a *first Pyramidal Triangular*, whose Root is $n = 1$ In. 232) viz. *AAA*: If $n = 2 = A$, *B*, the Number of all the *possible Ternaries* are 4 (a *first Pyramidal*, whose Root is $n = 2$) viz. *AAA*, *ABB*, *AAB*, *BBB*: If $n = 3 = A$, *B*, *C*, the Number of all the *possible Ternaries* are 10 (a *first Pyramidal*, whose Root is $n = 3$) viz. *AAA*, *AAB*, *AAC*, *ABB*, *ACC*, *ABC*, *BBC*, *BBB*, *BCC*, *CCC*: But this will always hold, ∴ The Number of all the *possible Ternaries*, in any Number of things, is a *first Pyramidal*, whose Root is n.

Con. 3. If $n = 1 = A$, the Number of all the possible *Quaternaries* is 1 (a *second Pyramidal Triangular*, whose Root is $n = 1$ In. 232.) viz. *AAAA*: If $n = 2 = A$, *B*, the Number of all the *possible Quaternaries* are 5 (a *second Pyramidal*, whose Root is $n = 2$) viz. *AAAA*, *AAAB*, *AABB*, *ABBB*, *BBBB*: If $n = 3 = A$, *B*, *C*, the Number of all the *possible Quaternaries* are 15 (a *second Pyramidal*, whose Root is $n = 3$) viz. *AAAA*, *AAAB*, *AAAC*, *AABB*, *AACC*, *AABC*, *ABBB*, *ABBC*, *ABCC*, *ACCC*, *BBBB*, *BBBC*, *BBCC*, *BCCC*, *CCCC*: If $n = 4 = A$, *B*, *C*, *D*, the Number of all the *possible Quaternaries* are 35 (a *second Pyramidal* Number, whose Root is $n = 4$) viz. *AAAA*, *AAAB*, *AAAC*, *AAAD*, *AABB*, *AABC*, *AABD*, *AACC*, *AACD*, *AADD*, *ABBB*, *ABBC*, *ABBD*, *ABCD*, *ACCB*, *ACCD*, *ADDD*, *ADDB*, *ADDC*, *BBBB*, *BBBC*, *BBBD*, *BBCC*, *BBDD*, *BBCD*, *BCCC*, *BCCD*, *BCDD*, *BDDD*,

BDDD, CCCC, CCCD, CCDD, CDDD, DDDD: But this will still hold, if $n=5$, $n=6$, $n=7$, *&c.* ∴ the Number of all the *possible Quaternaries*, that any given Number of things will admit of, is a *second Pyramidal Triangular*, whose Root is n.

Con. 4. And by the same Method it will be found, that the Number of all the *possible Quinaries*, that any Number of Things can admit of, will be a *third Pyramidal Triangular*; the Number of all the *possible Senaries*, a *fourth Pyramidal*; the Number of all the *possible Septenaries*, *a fifth Pyramidal*, &c. whose Root is n; whence we may infer the Certainty of this Theorem, *ad Infinitum*. Q. E. D.

Corollary LXIII.

239. If the Number of *Redundant Combinations* of any Kind be required alone, subtract the Number of *Simple Combinations* from the Number of *possible Ones*, and the Remainder will be the *Combinations* required.

Definition LXXX.

240. By *Permutation* is meant the Changes that any *Combination* of Things will admit of, in respect of their Order or Situation.

Partition XX.

241. *Permutations* are either of such Combinations as are *Simple*, or such as are *Redundant*.

Theorem XI.

242. The Number of all the *Permutations* that any *Simple Combination* admits of, is equal to the Product of all the Natural Numbers, beneath the Number of things combined, multiplied one into another; *i. e.* if the Combination be a *Pair* it is equal to 2×1; if a *Ternary* it is equal to $3\times2\times1$; if a *Quaternary* to $4\times3\times2\times1$; if a *Quinary* to $5\times4\times3\times2\times1$, &c. *ad Infinitum*. And universally all the *Permutations*, which the n Number of things combined will admit of, is equal to $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}$ *&c.* till the last Factor be equal to Unity.

Demonstration.

If $n=2=A, B$, the *Permutations* are $2\times1=2$, viz. *AB*, *BA*: If $n=3=BAC$, the Number of the *Permutations* are $3\times2\times1=6$, viz. *ABC*, *ACB*, *BAC*, *BCA*, *CBA*, *CAB*: If $n=4=A, B, C, D$, the *Permutations* are $4\times3\times2\times1=24$, viz. *ABCD*, *ABDC*, *ACBD*, *ACDB*, *ADBC*, *ADCB*, *BCDA*, *BCAD*, *BDCA*, *BDAC*, *BADC*, *BACD*, *CDBA*, *CDAB*, *CBAD*, *CBDA*, *CADB*, *CABD*, *DABC*, *DACB*, *DCBA*, *DCAB*, *DBAC*, *DBCA*. And in the same manner, the Number of all the Changes that any Quinary

Quinary admits of is found to be $5 \times 4 \times 3 \times 2 \times 1 = 120$; of a *Senary*, $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$; of a *Septenary* $720 \times 7 = 5040$: of an *Octonary* $5040 \times 8 = 40320$; &c. Q. E. D.

Corollary LXIV.

243. Hence we may learn to compute in what Time, at a certain Rate, all the Changes can be rung upon any proposed Number of different Bells: *Ex. gr.* All the Changes that can be rung upon a Peal of 7 Bells are $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$; which by allowing 15 Changes to a Minute, will require 336 Minutes, or 5 Hours 36 Minutes, in ringing out. All the Changes that can be rung upon a Peal of 10 Bells are $10 \times 9 \times 8 \times 7 \times 5040 = 3628800$, which by allowing 10 Changes to a Minute, will require 362880 Minutes, or 36 Weeks, continual ringing before the Peal be rung out. And the same may be easily applied to other Instruments of Musick.

Corollary LXV.

244. And hence also is easily deduced how to find all the possible Changes that can be rung upon any single tunable Set of Bells of any given Number; *i. e.* First, by Pairs; Secondly, by Ternaries; Thirdly, by Quaternaries, &c. *Ex. gr.* Suppose the given Number of Bells be $7 = n$.

1. The Number of Ones is 7, whose Permutations are $7 = n$.
2. The Number of Pairs in 7 Things is 21 (In. 236) and the Permutations of every Pair are 2 (In. 242) therefore the Permutations of all the Pairs in 7 different Things are $2 \times 21 = 42 = n \times \overline{n-1}$.
3. The Number of Ternaries in 7 different Things is 35 (In. 236) and the Permutations of every Ternary are 6 (In. 242) therefore the Permutations of all the Ternaries are $6 \times 35 = 210 = n \times \overline{n-1} \times \overline{n-2}$.
4. The Number of Quaternaries is 35 (In. 236) and the Permutations of every Quaternary are 24 (In. 242) therefore the Permutations of all the Quaternaries are $24 \times 35 = 840 = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$.
5. The Number of Quinaries is 21 (In. 236) and the Permutations of every Quinary are 120 (In. 242) therefore the Permutations of all the Quinaries are $120 \times 21 = 2520 = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}$.
6. The Number of Senaries is 7 (In. 236) and the Permutations of every Senary are 720 (In. 242) therefore the Permutations of all the Senaries are $720 \times 7 = 5040 = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}$.
7. The Number of Septenaries is 1 (In. 236) whose Permutations are $5040 = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \times \overline{n-6}$.

Therefore the Number of all the possible Changes, which can be rung upon a tuneable Peal of 7 Bells, is equal to $7 + 42 + 210 + 840 + 2520 + 5040 + 5040 = 13699$; which by allowing 15 Changes to a Minute will take up $913\frac{4}{15}$ Minutes (or 15 Hours 13 Minutes near) in ringing out.

The Theorem then for finding all the possible Changes, which a given Number n of different Things will admit of, is $n + n \times \overline{n-1} + n \times \overline{n-1} \times \overline{n-2} + n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} + n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}$, &c. till the last Multiplier be Unity.

Theorem XII.

245. The Number n of all the *Permutations*, which any *Redundant Combination* admits of, putting p for the Number of Times that the same Thing occurs in such Combination, is equal to the Number of *Permutations* found, as in the last Theorem, divided by 2×1 if $p = 2$, by $3 \times 2 \times 1$ if $p = 3$, by $4 \times 3 \times 2 \times 1$ if $p = 4$, &c. or universally by $p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \overline{p-4}$ &c. till the last Factor be equal to Unity; *i. e.* the Number of all the *Permutations* is equal to

$$\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \text{ \&c.}}{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \overline{p-4} \text{ \&c.}}$$

Demonstration.

If one Thing occur twice, *i. e.* if $p = 2$, then the Permutation of two Things A, A, will be $\frac{2 \times 1}{2 \times 1} = 1$, of three Things ($A$, A, B) will be $\frac{3 \times 2 \times 1}{2 \times 1} = 3$ (viz. *AAB*, *ABA*, *BAA*); of four Things (A, A, B, C) will be $\frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12$ (viz. *AABC*, *AACB*, *ABAC*, *ABCA*, *ACAB*, *ACBA*, *BAAC*, *BACA*, *BCAA*, *CAAB*, *CABA*, *CBAA*); of five Things (A, A, B, C, D) will be $\frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$ (viz. *AABCD*, *AABDC*, *AACBD*, &c.)

If one Thing occur thrice, *i. e.* if $p = 3$, then the Permutations of three Things (A, A, A) will be $\frac{3 \times 2 \times 1}{3 \times 2 \times 1} = 1$; of four Things ($A$, A, A, B) will be $\frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 4$ (viz. *AAAB*, *AABA*, *ABAA*, *BAAA*); of five Things (A, A, A, B, C) will be $\frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 20$ (viz. *AAABC*, *AAACB*, *AABAC*, *AABCA*, *AACAB*, *AACBA*, *ABAAC*, *ABACA*, *ABCAA*, *ACAAB*, *ACABA*, *ACBAA*, *BCAAA*, *BACAA*, *BAACA*, *BAAAC*, *CBAAA*, *CABAA*, *CAABA*, *CAAAB*); of six Things (A, A, A, B, C, D) will be $\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 120$ (viz. *AAABCD*, *AAABDC*, *AAACBD*, *AAACDB*, &c.)

But this Law will always hold, whatever n or p be. Therefore, &c. Q. E. D.

Corollary LXVI.

Corollary LXVI.

246. If one Thing occur p times, another q times, a third r times, *&c.* then the Number of *Permutations* in the n Number of things will be

$$\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \times \overline{n-6} \times \overline{n-7} \text{ \&c.}}{p \times \overline{p-1} \times \overline{p-2} \text{ \&c.} \times q \times \overline{q-1} \times \overline{q-2} \text{ \&c.} \times r \times \overline{r-1} \times \overline{r-2} \text{ \&c. \&c.}}$$

Theorem XIII.

247. If n be put for any Number of Things, then the Number of all the *Permutations* in all the Varieties of Things that such a Number can possibly admit of is $n^1+n^2+n^3+n^4$, *&c.* till the highest Power be n^n.

Demonstration.

If $n=2=A, B$, then the Permutations of the Simple Pairs are 2 (*viz.* *AB*, *BA* In. 242) to which if you add the Redundant Pairs *AA*, *BB*, the Sum will be $4=2^2$; therefore 2^1+2^2 is the Number of all the possible Permutations, that can happen to every Variety of Things, whose Number exceeds not *Two.*

If $n=3=A, B, C$, then the Permutations by Pairs are $9=3^2$ (*viz.* 6 by In. 242, with *AA*, *BB*, *CC*;) the Permutations by Ternaries are $27=3^3$ (*viz.* 6 by In. 242, with *AAA*, *BBB*, *CCC*; also the Permutations arising from these redundant Ternaries, *AAB*, *AAC*, *BBA*, *BBC*, *CCA*, *CCB* (which by In. 245 are $6\times3=18$) in all $6+3+18$); Therefore $3^1+3^2+3^3$ or $3+9+27=39$ is the Number of all the possible Permutations that can happen to every Variety of Things whose Number exceeds not *Three*.

And after the same manner the Number of all the possible Permutations that four Things can admit of will be found to be $4^1+4^2+4^3+4^4=340$; that five Things can admit of to be $5^1+5^2+5^3+5^4+5^5=3905$, *&c.* Whence we may conclude the Truth of the Theorem. Q. E. D.

Corollary LXVII.

243. Hence we may learn to compute the Time, at a given Rate, wherein all the possible Changes (or all the possible different Noises) that can be made upon Bells not exceeding a given Number; *Ex. gr.* all the possible Changes that can be rung upon 7 Bells are $7+7^2+7^3+7^4+7^5+7^6+7^7=960799$, which, by allowing 15 Changes to a Minute, will require upwards of 6 Weeks, 2 Days, 11 Hours, 33 Minutes continual ringing, before all the different Changes can be rung out.

And all the Changes that can be rung upon 10 Bells are $10^1+10^2+10^3+10^4+10^5+10^6+10^7+10^8+10^9+10^{10}=11111111110$, which, by allowing 10 Changes to a Minute, would require upwards of 2113 Years and 51 Weeks continual ringing Night and Day, before all the different Changes cou'd

cou'd be rung out. So great a Difference is there upon the Addition of three Bells. And if but one Bell more were added (*i. e.* if the Number of Bells were 11) the Number of Changes wou'd be 31384283767I, which, by allowing 10 Changes again to a Minute, wou'd take up above 59711 Years, 18 Weeks, and 12 Hours in ringing out.

Corollary LXVIII.

249. Hence laſtly we may learn the exceeding Abſurdity of that Notion, which ſome wou'd have us believe, that all Things in the Courſe of Providence do happen by Chance; *i. e.* without any Deſign: For it is plain that no two Things only, much more an Infinity of Things, can, for any Time, conſpire exactly to the ſame End: Becauſe the Proportion of miſſing to that of hitting is as Infinity to one; *i. e.* there is no Proportion at all; and therefore it can never happen at all. (In. 36.)

The End of the firſt PART.

ARITHMETICAL INSTITUTIONS.

PART II.

Of the Algorism of NUMBERS.

CHAP. I.

General DEFINITIONS.

DEFINITION I.

250. NUMERAL *Algorism* is the Method of handling Numbers, according to their *Notation.*

PARTITION I.

251. And this is either *Natural* or *Artificial.*

DEFINITION II.

252. *Natural Algorism* is that which proceeds by the *Natural Numbers.*

DEFINITION III.

253. *Artificial Algorism* is that which proceeds by *Artificial Numbers*, substituted in the Room of *Natural* ones, called *Logarithms.*

CHAP. II.

Of the Notation of Natural NUMBERS.

DEFINITION IV.

254. THE *Notation* of Numbers is the Method of expressing them by proper Names and Characters, according as they are above or below Unity.

PARTITION II.

255. *Notation* is either of *Integers*, or *Fractions*.

HYPOTHESIS I.

256. The general received Way of *Naming* or *Denominating Integers* proceeds by *Tens* as follows.

The *first Denomination* is that of *Units*, whose particular Names and Characters are,

One,	*Two*,	*Three*,	*Four*,	*Five*,	*Six*,	*Seven*,	*Eight*,	*Nine*,
1,	2,	3,	4,	5,	6,	7,	8,	9,

called *Figures*, or *Digits*.

The *second Denomination* is of *Tens*, or ten Units, which are signified by the Character *o*, called a *Cypher*, set to the Right-hand of the foregoing Characters; and are particularly named thus,

Ten,	*Twenty*,	*Thirty*,	*Forty*,	*Fifty*,	*Sixty*,	*Seventy*,	*Eighty*,	*Ninety*.
10,	20,	30,	40,	50,	60,	70,	80,	90.

The *third Denomination* is of *Hundreds*, or *Ten Tens*, which are signified by the Character *o* set again to the Right-hand of the *Denomination* of *Tens*, thus; 100, *i. e.* 1 Hundred; 200, *i. e.* 2 Hundred; 300, *i. e.* 3 Hundred; 400, &c.

The *fourth Denomination* is of *Thousands* or *Ten Hundreds*, characteriz'd with three *Cyphers*; 1000, 2000, 3000, 4000, 5000, &c.

The *fifth Denomination* is of *Ten Thousands*, with four *Cyphers*; 10000, 20000, 30000, &c.

The *sixth Denomination* is of *Hundred Thousands*; the *seventh*, of *Millions* or *Ten Hundred Thousands*; the *eighth*, of *Ten Millions*; the *ninth*, of *Hundred Millions*, &c. *ad Infinitum*, each following *Denomination* still consisting of one *Cypher* or *Place* more than the foregoing one.

DEFINITION V.

257. The Character *o*, is also stiled *Nothing*, because it signifies nothing of it self, but only serves to distinguish the *Places* of *Figures* according to their proper

proper Denominations: So that if a Number conſiſt of different Denominations, each *Figure* obtains its proper *Place* in Order together with the Reſt. *Ex. gr.*

15		10 + 5 or Fifteen
530		500 + 30
1786		1000 + 700 + 80 + 6
68054	denotes	60000 + 8000 + 50 + 4
130507		130000 + 500 + 7
9100753		9000000 + 100000 + 700 + 50 + 3
321321321		321000000 + 321000 + 321

Hypothesis II.

258. And for the ready Diſcovery of the Value of each *Figure*, in Numbers which conſiſt of many *Figures*, every *three Places* from *Units Place*, are diſtinguiſhed into *Periods*, the loweſt of which to the Right hand in Integers is ſtiled the *Period of Units*; the ſecond, of *Thouſands*; the third, of *Millions*; the fourth, of *Billions* or *ten Hundred Millions*; the fifth, of *Trillions* or ten *Hundred Billions*; the ſixth of *Quadrillions*, or ten *Hundred Trillions*, &c. to any Number aſſignable; *Ex. gr.* the following is a Number of eighteen Places or ſix Periods $\overline{654}\,\overline{654}\,\overline{654}\,\overline{654}\,\overline{654}\,\overline{654}$ which is thus read 654 *Quadrillions*, 654 *Trillions*, 654 *Billions*, 654 *Millions*, 654 *Thouſands*, 654.

Scholium I.

259. This Method of characterizing and diſtinguiſhing Numbers, is deſervedly reckoned to be one of the moſt wonderful Diſcoveries that ever human Sagacity attain'd to. By this you ſee that Numbers, which would take up even Millions of Ages in being enumerated one by one, are diſtinctly expreſſed, and handled with the ſame Exactneſs, as the ſmalleſt: A Performance which, if conſtant Uſe did not render it familiar and eaſy, we wou'd hardly believe poſſible.

The Invention of *Numeral Figures* is attributed to the *Indians*, who taught them to the *Arabs*, and they to the *Moors*, with whom it came into *Spain*; as the learned Dr. *Wallis* thinks, between the 9th and 10th Centuries.

Hypothesis III. And Partition III.

260. If a *Point* be ſet to the Right-hand of *Units Place*, a *Figure* ſet to the Right-hand of that *Point* is put for ſo many *tenth Parts*; and a *Figure* to the Right-hand of that again, for ſo many *Thouſand Parts*; and ſo on to *Ten Thouſandths*, *Hundred Thouſandths*, *Millioneths*, &c. And theſe Parts are called *Decimal Fractions*, as thoſe are called *Vulgar Fractions* (In. 88 above); theſe having

having their Denominators always underſtood, and not expreſſed, as the others have; *Ex. gr.*

$$\left.\begin{array}{r} 0.5 \\ 0.05 \\ 0.005 \\ 0.0005 \\ 0.00005 \end{array}\right\} \text{ is } \left\{\begin{array}{c} \frac{5}{10} \\ \frac{5}{100} \\ \frac{5}{1000} \\ \frac{5}{10000} \\ \frac{5}{100000} \end{array}\right\} \textit{ i. e. } \left\{\begin{array}{l} 5 \textit{ Tenths.} \\ 5 \textit{ Hundredths.} \\ 5 \textit{ Thouſandths.} \\ 5 \textit{ Ten Thouſandths.} \\ 5 \textit{ Hundred Thouſandths.} \end{array}\right.$$

&c. &c. &c.

Partition IV.

261. *Decimal Fractions* are either *Exact* or *Approximant.*

Definition VI.

262. An *Exact Decimal Fraction* is that which expreſſes the Ratio of the Part to the Whole exactly; as $0.4 = \frac{4}{10} = \frac{2}{5}$, which expreſſes the Ratio of the Part 4 to its Whole 10; or, which is the ſame thing, of the Part 2, to its Whole 5. An *Approximant Decimal Fraction* is that which does not ſo, but approaches infinitely near it; an Example of which ſee (In. 318.)

Scholium II.

263. The firſt Appearance of *Decimal Fractions* ſeems to be in a Work writ by *Johannes Mullerus Regiomontanus*, about the Year 1464 (*Wallis's Algebra*, Chap. 9.)

Partition V.

264. Numbers both Integer and Fracted are divided into *Monomes*, and *Polynomes*.

Definition VII.

265. By *Monomes* are meant ſuch Numbers as conſiſt but of one ſignificant Figure; as, 1, 2, 5, 60, 0.007, 9000, *&c.*

Partition VI.

266. *Monomes* are *Homogeneous*, when they are of the ſame Denomination; as 2 and 4, 50 and 60, 700 and 900, 3 and 8, *&c.* *Heterogeneous*, when they are of different Denominations; as, 2 and 20, 3 and 0.3; 50 and 400, *&c.*

Definition VIII.

267. *Polynomes* are such Numbers as consist of more *Figures* than one; and these are called *Binomials*, when they consist but of 2 *Figures*; *Trinomials*, when they consist of 3 *Figures*; *Quadrinomials*, of 4, &c.

Partition VII.

268. Again a *Polynome* is said to be *Pure*, when it expresses the same Species of Quantity according to the received way of Notation; and *mixed*, when not: Of the former Kind are the Numbers (In. 257); for the latter see Chap. III. following.

Definition IX.

269. *Polynomes* are said to be of the same Denomination when they consist of the same Number of Places, each of the same Denomination.

Corollary I.

270. *Cyphers* to the Right-hand of Integral Figures do encrease them in a ten-fold Ratio; and to the Left-hand of Fractional Figures decrease them in the same Ratio.

Corollary II.

271. Cyphers to the Left-hand of Integers, and to the Right-hand of Fractions signify nothing, unless it be to distinguish their Places from Unity in respect of other Numbers.

Corollary III.

272. Only Homogeneous Figures, or those of the same Denomination, can be added to and subtracted from one another (In. 67.)

Corollary IV.

273. All the Figures in a Fractional Polynome may be read with the Denomination of the lowest; *Ex. gr.*

$$\left.\begin{array}{l}0.25\\0.325\\0.03072\end{array}\right\}\text{ is read }\left\{\begin{array}{l}25\ \textit{Hundreds.}\\325\ \textit{Thousandths.}\\3072\ \textit{Hundred Thousandths.}\end{array}\right.$$

Corollary V.

274. The least Monome of any Place from Unity, is when it is expressed by the Figure 1, as 1, 10, 100, 1000, &c. .1 .01 .001 .0001 &c. the when it is expressed by 9; as 9, 90, 900 and .9 .09 .009 &c.

Corollary VI.

275. The greatest Polynome of any Number of Places is when all its Places are filled up with 9s; as 99, 999, 9999, &c.

Corollary VII.

276. The least Integral Monome of any Number of Places is greater by Unity than the greatest Integral Polynome of one Place fewer; *i. e.* 10 is greater by one than 9, 100 than 99, 1000 than 999, 10000 than 9999, &c.

Corollary VIII.

277. If a Figure be removed from its Place to the Place next higher, it will be encreased by as many 9s as it contains Units; and if to the Place next lower it will be so many 9s decreased; *Ex. gr.* $7000 = 700 \times 9 + 700 : 700 = 70 \times 9 + 70 : 70 = 7 \times 9 + 7 : 7 = 0.7 \times 9 + 0.7$: And therefore

Corollary IX.

278. If the two lowest Figures in a Polynome be considered as so many Units, there will be just as many 9s omitted, except the first Figure to the Left-hand: *Ex. gr.* in 12 there is omitted 1×9; *i. e.* $1 + 2 = 12 - 1 \times 9$: in 47, there is omitted 4×9; *i. e.* $4 + 7 = 47 - 4 \times 9$: in 239, there is omitted 23×9; i. *e.* $23 + 9 = 239 - 23 \times 9$: in 50308, there is omitted 5030×9; *i. e.* $5030 + 8 = 50308 - 5030 \times 9$, &c.

Corollary X.

279. In Multiplication when the Multiplier is 10, 100, 1000, &c. the Product is had, by only advancing all the Figures in the Multiplicand as many Places higher, as there are Places above Units Place in the Multiplier; and when the Multiplier is 0.1, 0.01, 0.001, &c. by depressing them as many Places lower as there are Places below Units Place in the Multiplier. *Ex. gr.* $36 \times 1000 = 36000$; $36 \times 0.001 = 0.036$. And on the contrary

Corollary XI.

280. In Division when the Divisor is 10, 100, 1000, &c. the Quotient is had, by only depressing all the Figures in the Dividend as many Places lower, as there are Places above Units Place in the Divisor; and when the Divisor is 0.1, 0.01, 0.001, &c. by advancing them as many Places higher as there are Places below Units Place in the Divisor (In. 82, and 115.)

Corollary XII.

281. If the Product of the two highest Figures alone, or together with the Increase they receive from the lower Figures, in any two Integral Factors do consist

consist of two Places, then the Product made by those entire Factors will consist of as many Places as are in both Factors; otherwise of one Place less. Or, which is the same thing, if the highest Figure of the Product be lesser than the highest Figure of either Factor, then will the Product consist of as many Places as are in both the Factors; otherwise of one Place less. *Ex. gr.* $800 \times 70 = 56000$; $20 \times 40 = 800$.

Corollary XIII.

282. How much soever any Multiplying Figure is above or below Units Place, so much will it advance or depress the Multiplicand above or below Units Place. Whence

Corollary XIV.

283. In Multiplication there will always be as many either Cyphers or Fractional Places in the Product, as there are in both the Factors; Consequently,

Corollary XV.

284. In Division the Quotient together with the Divisor will make up as many, either Cyphers or Fractional Places between them, as are contained in the Dividend.

Corollary XVI.

285. How much soever any Divisor is of a lower Denomination than the Dividend, so much higher will the Quotient be advanced above Units Place: and, *vice versâ*, how much soever higher the Denomination of a Divisor is above that of the Dividend, so much the lower will the Quotient Figure be depressed below Unity, Therefore

Corollary XVII.

286. Whenever the highest Figures of the Dividend are greater than those of the Divisor, if the latter be set directly under the former, as in these, $\frac{847}{24}$, $\frac{7200}{635}$; And whenever they are less, if the highest Figure in the Divisor be set under the second of the Dividend, as in these $\frac{144}{88}$, $\frac{144}{16}$; then it is manifest that the highest Figure in the Quotient will always be of the same Denomination with that Place in the Dividend, which stands over Unity in the Divisor. *Ex. gr.* $\frac{12000}{30} = 400$. $\frac{12.000}{3.000} = 0.004$. $\frac{0.120}{0.03} = 4$. $\frac{0.12}{00.2} = 0.4$ &c.

Corollary XVIII.

Corollary XVIII.

287. The leaſt Integral Monomes of their Number of Places has been ſhewn to be $10=9+1$, $100=99+1$, $1000=999+1$, $10000=9999+1$ *&c.* (In. 271) whoſe Powers are as follows (In. 280.)

Roots	10	100	1000	&c.
Squares	100	10000	1000000	&c.
Cubes	1000	1000000	1000000000	&c.
Biquadrates	10000	100000000	1000000000000	&c.
&c.	&c.	&c.	&c.	

Whence it appears by Inſpection, that counting from Unit's Place, there can be but one Place in the Root for every Period of two Places in the *Square*, of three Places in the *Cube*, of four in the *Biquadrate*, &c. and one Place for what remains over and above; which in each Power is alſo to be taken as a Period of it ſelf.

Hypothesis IV.

288. The Periods of Powers are diſtinguiſhed by drawing Lines over the Head of each; *Ex, gr.* the Number 68719476736 if conſidered as a *Square* conſiſts of ſix Periods $\overline{6}\ \overline{87}\ \overline{19}\ \overline{47}\ \overline{67}\ \overline{36}$, if as a *Cube* of four Periods $\overline{68}\ \overline{719}\ \overline{476}\ \overline{736}$, if as a *Biquadrate* of three $\overline{687}\ \overline{1947}\ \overline{6736}$, if as a *ffrſt Surſolid* of three $\overline{6}\ \overline{87194}\ \overline{76736}$, if as a ſixth Power of two $\overline{68719}\ \overline{476736}$, *&c.*

Corollary XIX.

289. For every Cypher, or Fractional Place poſſeſſing the Right-hand in any Number conſidered as a *Root*, there muſt be two in its Square, three in its Cube, four in its Biquadrate, *&c.*

Corollary XX.

290. The Root of the next Homologous Power to the higheſt Period in every given Reſolvend is the firſt Figure of the Root required, and of the ſame Place from Unity that the higheſt Period is from the Period of Units. *Ex. gr.* in the Square Reſolvend $\overline{6}\ \overline{87}\ \overline{19}\ \overline{47}\ \overline{67}\ \overline{36}$ the Periods are ſix, *viz.* $60000000000 + 8700000000 + 19000000 + 470000 + 6700 + 36$, conſequently the Root conſiſts of ſix Places (In 284) *i. e.* if the loweſt Place in the loweſt Period be Units Place, then the higheſt Place in the Root is the Place of Hundred Thouſands (In. 254) and becauſe by the Table (In. 349) the next greateſt Square under 60000000000 is 40000000000 whoſe Root is 200000, therefore 200000 is the firſt Figure in the Root. Again in the Cube Reſolvend $\overline{68}\ \overline{719}\ \overline{476}\ \overline{736}$ the Periods are four, *viz.* $68000000000 + 719000000 + 476000 + 736$ conſequently the Root conſiſts of four Places (In. 284) *i. e.*

if

if the lowest Place in the lowest Period be Units Place, then the highest Place in the Root is the Place of Thousands (In. 254) and because by the Table (In. 349) the next greatest Cube under 68000000000 is 64000000000 whose Root is 4000, therefore 4000 is the first Figure in the Root. In like manner in the Biquadrate Resolvend $\overline{687}\ \overline{1947}\ \overline{6736}$ the Periods are three, *viz.* 68700000000 + 194,0000 + 6736, consequently the Root consists of three Places (In.. 284) *i. e.* if the lowest Place in the lowest Period be Units Place, then the highest Place in the Root is the Place of Hundreds (In. 254) and because by the Table (In. 349) the next greatest Biquadrate under 68700000000 is 62500000000 whose Root is 500, therefore 500 is the first Figure in the Root.

CHAP. III.

Of Mixed Polynomes, and the Tables of Weight, Coin, and MEASURES.

DEFINITION X.

291. I Call that a mixed *Polynome* which expresses different Denominations of Quantity according to the Table of Coin, Weights, Measures, *&c.* to which it belongs.

DEFINITION XI.

292. The Standard of the currant Coin of *England* is called *Sterling* or *Esterling*, by which is meant the Weight of an *English* Silver Penny, which ought to be equivalent in Weight with 32 Grains of well dried Wheat taken out of the middle of the Ear: The fourth Part of this is termed a *Farthing* marked *qr.* for *Quadrans*, which is the least Denomination of *English* Coin; as in the following

TABLE.

4 *qr* = 1*d* *Denarius*, a Penny
48 *qr* = 12*d* = 1*s* *Solidus*, a Shilling
960 *qr* = 240*d.* = 20*s* = 1*l* *Libra*, a Pound Sterling

05*s.* 00*d*	is	*A Crown.*
10*s.* 00*d*	is	*An Angel.*
06*s.* 80*d*	is	*A Noble.*
13*s.* 04*d*	is	*A Mark.*
1*l.* 01*s.* 00*d*	is	*A Guinea.*

D There-

Therefore

If 1 *l.* be the Integer.

s.	*d.*	*l.*	*s.*	*d.*	*l.*	*d.*	*qr.*	*l.*
10 .	0 =	$\frac{1}{2}$	02 .	0 =	$\frac{1}{10}$	7 .	2 =	$\frac{1}{32}$
06 .	8 =	$\frac{1}{3}$	01 .	8 =	$\frac{1}{12}$	6 .	0 =	$\frac{1}{40}$
05 .	0 =	$\frac{1}{4}$	01 .	4 =	$\frac{1}{15}$	5 .	0 =	$\frac{1}{48}$
04 .	0 =	$\frac{1}{5}$	01 .	3 =	$\frac{1}{16}$	4 .	0 =	$\frac{1}{60}$
03 .	4 =	$\frac{1}{6}$	00 .	0 =	$\frac{1}{20}$	3 .	3 =	$\frac{1}{64}$
2 .	6 =	$\frac{1}{8}$	00 .	10 =	$\frac{1}{24}$	3 .	0 =	$\frac{1}{80}$
			00 .	08 =	$\frac{1}{30}$		&c.	

If 1 *s.* be the Integer.

d.	*qr.*	*s.*
6 .	0 =	$\frac{1}{2}$
4 .	0 =	$\frac{1}{3}$
3 .	0 =	$\frac{1}{4}$
2 .	0 =	$\frac{1}{6}$
1 .	2 =	$\frac{1}{8}$
1 .	0 =	$\frac{1}{12}$ &c.

DEFINITION XII.

293. *Troy Weight* is that whereby *Corn*, *Bread*, *Liquors*, *Gold*, *Silver*, *Jewels*, *Amber*, *Electuaries*, &c. are weighed: the least Denomination of this Weight is an *Artificial Grain*, *Gr.* 24 of which are equal in Weight to 32 Grains of Wheat.

The TABLE.

24 *Gr.* = 1 *P. W. Penny Weight*
480 *Gr.* = 20 *P. W.* = 1 *Oz. Ounce*
5760 *Gr.* = 240 *P. W.* = 12 *Oz.* = 1 *lb. Pound Troy*

24 *Blanks* = 1 *Periot.*
20 *Periots* = 1 *Droite.*
24 *Droites* = 1 *Mite.*
20 *Mites* = 1 *Grain*, &c. as before.

Therefore

If 1 *lb.* be the Integer.

oz.	*p.w.*	*lb.*	*oz.*	*p. w.*	*lb.*
6 .	0 =	$\frac{1}{2}$	1 .	4 =	$\frac{1}{10}$
4 .	0 =	$\frac{1}{3}$	1 .	0 =	$\frac{1}{12}$
3 .	0 =	$\frac{1}{4}$	0 .	12 =	$\frac{1}{20}$
2 .	8 =	$\frac{1}{5}$	0 .	8 =	$\frac{1}{30}$
2 .	0 =	$\frac{1}{6}$	0 .	6 =	$\frac{1}{40}$
1 .	12 =	$\frac{1}{8}$		&c.	&c.

If 1 *oz.* be the Integer.

p. w.	*gr.*	*oz.*	*p.w.*	*gr.*	*oz.*
10 .	0 =	$\frac{1}{2}$	2 .	0 =	$\frac{1}{10}$
6 .	16 =	$\frac{1}{3}$	1 .	6 =	$\frac{1}{12}$
5 .	0 =	$\frac{1}{4}$	1 .	0 =	$\frac{1}{20}$
4 .	0 =	$\frac{1}{5}$	0 .	16 =	$\frac{1}{30}$
3 .	8 =	$\frac{1}{6}$	0 .	12 =	$\frac{1}{40}$
2 .	12 =	$\frac{1}{8}$		&c.	&c.

DEFINITION XIII.

294. *Apothecaries Weight* is a different Table of *Troy Weight*, by which Apothecaries compound their Medicines.

The TABLE.

20 *Gr.* = 1℈ *Scruple.*
60 *Gr.* = 3℈ = 1ʒ *Drachm.*
480 *Gr.* = 24℈ = 8ʒ = 1℥ *Ounce*
5760 *Gr.* = 288℈ = 96ʒ = 12℥ = 1℔ *Troy*, as above.

Three-

Therefore

If 1*lb.* be the Integer.

℥	ʒ	℔	℥	ʒ	℈	gr.	℔	℥	ʒ	℈	gr.	℔	℥	ʒ	℈	gr.	℔
06	0	= $\frac{1}{2}$	02	3	0	12	= $\frac{1}{5}$	1	1	1	16	= $\frac{1}{10}$	0	3	0	12	= $\frac{1}{30}$
04	0	= $\frac{1}{3}$	02	0	0	0	= $\frac{1}{6}$	1	0	0	00	= $\frac{1}{12}$	0	2	1	04	= $\frac{1}{40}$
03	0	= $\frac{1}{4}$	1	4	0	0	= $\frac{1}{8}$	0	4	2	08	= $\frac{1}{20}$		*&c.*			*&c.*

If 1 ℥ be the Integer.

ʒ	℈	gr.	℥	ʒ	℈	gr.	℥	ʒ	℈	gr.	℥	ʒ	℈	gr.	℥
4	0	0	= $\frac{1}{2}$	1	1	16	= $\frac{1}{5}$	0	2	8	= $\frac{1}{10}$	0	0	16	= $\frac{1}{30}$
2	2	0	= $\frac{1}{3}$	1	1	00	= $\frac{1}{6}$	0	2	0	= $\frac{1}{12}$	0	0	12	= $\frac{1}{40}$
2	0	0	= $\frac{1}{4}$	1	0	00	= $\frac{1}{8}$	0	1	4	= $\frac{1}{20}$	*&c.*			*&c.*

Definition XIV.

295. *Averdupois Weight* is that whereby *Flesh*, *Butter*, *Cheese*, *Tallow*, *Salt*, *Flax*, *Hemp*, *Wax*, *Pitch*, *Tar*, *Rosin*, *Copper*, *Tin*, *Steel*, *Iron*, *Lead*, *Tobacco*, &c. and in general all Kinds of *Grocery Wares*, and whatsoever is subject to waste, are weighed. The Pound *Averdupois* Mr. *John Ward* of *Chester*, says, he found by a very nice Experiment to be equal to 14 *oz.* 11 *p.w.* 15$\frac{1}{2}$ *Grains Troy.* The least Denomination of this Weight is a *Dram*, *Dr.*

The TABLE.

	lb.
16 *Dr.* = 1 *Oz. Ounce.*	14 = a *Stone.*
256 *Dr.* = 16 *Oz.* = 1 *lb. Pound.*	28 = $\frac{1}{4}$ of *C.*
28672 *Dr.* = 1792 *Oz.* = 112 *lb.* = 1 *C. Hundred.*	56 = $\frac{1}{2}$ of *C.*
573440 *Dr.* = 35840 *Oz.* = 2240 *lb.* = 20 *C.* = 1 *Tun*	84 = $\frac{3}{4}$ of *C.*

Therefore

If 1 *C.* be the Integer,				If 1 *lb.* be the Integer.			
lb.	*C.*	*lb.*	*C.*	*Oz.*	*lb.*	*Oz.*	*lb.*
56	= $\frac{1}{2}$	14	= $\frac{1}{8}$	8	= $\frac{1}{2}$	1	= $\frac{1}{16}$
28	= $\frac{1}{4}$	7	= $\frac{1}{16}$	4	= $\frac{1}{4}$		*&c.*
16	= $\frac{1}{7}$		*&c.*	2	= $\frac{1}{8}$		

Definition XV.

296. The Denominations of *Wool-Weight* are as follows, the least Denomination being 1 *lb. Averdupois.*

The TABLE.

7 *lb.* = 1 *Cl. Clove.*
14 *lb.* = 2 *Cl.* = 1 *St. Stone.*
28 *lb.* = 4 *Cl.* = 2 *St.* = 1 *Tdd. Todd.*

182 *lb*

182 lb. = *26 Cl.* = *13 St.* = $6\frac{1}{2}$ *Tdd.* = *1 W. Wey.*
364 lb. = *52 Cl.* = *26 St.* = *13 Tdd.* = *2 W.* = *1 S. Sack.*
4368 lb. = *624 Cl.* = *312 St.* = *156 Tdd.* = *24 W.* = *12 S.* = *1 Laſt* or *Wey.*

Definition XVI.

297. The leaſt Denomination of *Long Meaſure* is an Inch *In.* which is ſuppoſed to be equal in Length to three Corns of Barly well dried, and taken out of the middle of the Ear.

The TABLE.

In.													
$2\frac{1}{4}$ =	1 *Nail.*												
3 =	$1\frac{1}{3}$ =	1 *Palm.*											
4 =	$1\frac{7}{9}$ =	$1\frac{1}{3}$ =	1 *Hand.*										
9 =	4 =	3 =	$2\frac{1}{4}$ =	1 *Span.*									
12 =	$5\frac{1}{3}$ =	4 =	3 =	$1\frac{1}{3}$ =	1 *Foot.*								
18 =	8 =	6 =	$4\frac{1}{2}$ =	2 =	$1\frac{1}{2}$ =	1 *Cubit.*							
36 =	16 =	12 =	9 =	4 =	3 =	2 =	1 *Yard.*						
45 =	20 =	15 =	$11\frac{1}{4}$ =	5 =	$3\frac{3}{4}$ =	$2\frac{1}{2}$ =	$1\frac{1}{4}$ =	1 *Ell.*					
60 =	$26\frac{2}{3}$ =	20 =	15 =	$6\frac{2}{3}$ =	5 =	$3\frac{1}{3}$ =	$1\frac{2}{3}$ =	$1\frac{1}{3}$ =	1 *Pace.*				
72 =	32 =	24 =	18 =	8 =	6 =	4 =	2 =	$1\frac{3}{5}$ =	$1\frac{1}{5}$ =	1 *Fathom.*			
198 =	88 =	66 =	$49\frac{1}{2}$ =	22 =	$16\frac{1}{2}$ =	11 =	$5\frac{1}{2}$ =	$4\frac{2}{5}$ =	$3\frac{3}{10}$ =	$2\frac{3}{4}$ =	1 *Perch.*		
7920 =	3520 =	2640 =	1980 =	880 =	660 =	440 =	220 =	176 =	132 =	110 =	40 =	1 *Furl.*	

8 *Furlongs*	make	1 *Mile.*
3 *Miles*	make	1 *League,*
$23\frac{4}{27}$ *Leagues*	make	1 *Degree,*

Or the $\frac{1}{360}$ Part of the Earth's Circumference.

Definition XVII.

298. The Diviſions of a Circle are as follows.

Every	*Circle*	is divided into	12 *S. Signs,*
	Sign		30° *Degrees,*
	Degree		60′ *Minutes,*
	Minute		60″ *Seconds,*
	Second		60‴ *Thirds,*
	Third		60⁗ *Fourths,*
	&c.		&c.

Therefore

″	′	°		″	′	°	S.	
60 =	1			108000 =	1800 =	30 =	1	
3600 =	60 =	1 S		1296000 =	21600 =	360 =	12 =	1 *Circle.*

Corollary XXI.

Corollary XXI.

299. The Circumference of the Earth upon this Suppoſition equals $69\frac{4}{9} \times 360^{\circ} = 25000$ Miles nearly; as was found by the learned *Picard*, and after him by the famous *Coſſini*; which is but 20 Miles leſs than what was found by our Country-man Mr. *Norwood* upon a leſs advantagious Experiment.

Definition XVIII.

300. The leaſt Denomination of *Superficial Meaſure* is a *Square Inch In.*; i. e. an Inch in Length and Breadth.

The Table *of* Superficial Meaſure.

In.						
144 =	1 *Foot Square.*					
1296 =	9 *F.* =	1 *Yard Square.*				
3600 =	25 =	$2\frac{7}{9}$ =	1 *Pace Square.*			
39204 =	$272\frac{1}{4}$ =	$30\frac{1}{4}$ =	$10\frac{89}{100}$ =	1 *Perch*, or *Pole Square.*		
1568160 =	10890 =	1210 =	$435\frac{6}{10}$ =	40 =	1 *Rood.*	
6272640 =	43560 =	4840 =	$1742\frac{4}{10}$ =	160 =	4 =	1 *Statute Acre.*

Definition XIX.

301. Becauſe an *Acre* or 160 Square *Perches* = 43560 Square *Feet* is equal to 40 *Perches* or $40 \times 16\frac{1}{2} = 660$ *Feet* in Length, and 4 *Perches* or $4 \times 16\frac{1}{2} = 66$ *Feet* in Breadth: Therefore in *Land Meaſure* there is commonly uſed a *Chain* of 66 *Feet* or 22 *Yards* Long; every 10 of which in Length and 1 in Breadth (*i. e.* every 10 Square *Chains*) makes an *Acre*. And this *Chain* is ſubdivided into 100 *Links* of 7.92 *Inches* each, according to which

Sq. Chains,	*Sq. Links,*	*Acre,*	*Roods,*	*Perches Square.*
10.000 =	10000 =	1 =	4 =	160
2.500 =	2500 =	$\frac{1}{4}$ =	1 =	40
0.0625 =	62.5 =	$\frac{1}{160}$ =	$\frac{1}{40}$ =	1.

Definition XX.

302. *Meaſures of Capacity* are ſuch as are of three Dimenſions; viz. *Length, Breadth*, and *Thickneſs, Height* or *Depth*: Of which there are eſpecially four Sorts uſed in *Britain*, viz. *Corn, Beer, Ale*, and *Wine Meaſure*: The leaſt Denomination to all which is called a *Cubic Inch*, i. e. an *Inch* in *Length, Breadth*, and *Thickneſs*.

Definition XXI.

303. In the Table of *Corn Meaſure*, 1 *Gallon* = 268.8 *Cubic Inches*, and the leaſt Denomination is a *Pint*, *Pt.*

The

The TABLE.

Pints.
2= 1 *Quart.*
8= 4= 1 *Gallon.*
16= 8= 2= 1 *Peck.*
64= 32= 8= 4= 1 *Bushel Winchester.*
128= 64= 16= 8= 2= 1 *Strike.*
256= 128= 32= 16= 4= 2= 1 *Carnock* or *Comb.*
512= 256= 64= 32= 8= 4= 2= 1 *Seam, Rast,* or *Quarter.*
3072=1536=384=192=48=24=12= 6= 1 *Wey.*
5120=2560=640=320=80=40=20=10=$1\frac{2}{3}$=1 *Last.*

DEFINITION XXII.

304. In *Beer Measure* 1 *Gallon* = 282 *Cubic Inches.*

The TABLE.

Gills.
2= 1 *Pint.*
4= 2= 1 *Quart.*
16= 8= 4= 1 *Gallon.*
144= 72= 36= 9=1 *Firkin.*
288=144= 72=18=2=1 *Kilderkin.*
576=288=144=36=4=2=1 *Barrel.*
864=432=216=54=6=3=$1\frac{1}{2}$=1 *Hogshead.*

DEFINITION XXIII.

305. In *Ale Measure* 1 *Gallon* = 282 *Cubic Inches.*

The TABLE.

Gills.
2= 1 *Pint.*
4= 2= 1 *Quart.*
16= 8= 4= 1 *Gallon.*
128= 64= 32= 8=1 *Firkin.*
256=128= 64=16=2=1 *Kilderkin.*
512=256=128=32=4=2=1 *Barrel.*
768=384=192=48=6=3=$1\frac{1}{2}$=1 *Hogshead.*

DEFINITION XXIV.

306. In *Wine Measure* the *Gallon* equals 231 *Cubic Inches*, by which are measured all *Wines*, *Brandies*, *Spirits*, *Mead*, *Perry*, *Cyder*, *Vinegar*, *Oyl*, *Honey*, &c.

The

The TABLE.

Gills, *Pints*, *Quarts*.

$16 = 8 = 4 = 1$ *Gallon* as before.

$18 = 1$ *Rundlet*.

$31\frac{1}{2} = 1\frac{3}{4} = 1$ *Barrel*.

$42 = 2\frac{1}{3} = 1\frac{1}{3} = 1$ *Tierce*.

$63 = 3\frac{1}{2} = 2 = 1\frac{1}{2} = 1$ *Hogshead*.

$84 = 4\frac{2}{3} = 2\frac{2}{3} = 2 = 1\frac{1}{3} = 1$ *Puntion*.

$126 = 7 = 4 = 3 = 2 = 1\frac{1}{2} = 1$ *Butt*.

$252 = 14 = 8 = 6 = 4 = 3 = 2 = 1$ *Tun*.

DEFINITION XXV.

307. The least Denomination of any Part of *Time* is a *Second*, from whence is formed the following Table of *Time*:

″	′			
60=	1 *Minute*.			
3600=	60=	1 *Hour*.		
86400=	1440=	24=	1 *Natural Day*.	
604800=	10080=	168=	7=	1 *Week*.

$\overset{\prime\prime}{2360587} = \overset{\prime}{39343} + \overset{\prime\prime}{7} = \overset{H.}{655} + \overset{\prime}{43} + \overset{\prime\prime}{7} = \overset{D.}{27} + \overset{H.}{7} + \overset{\prime}{43} + \overset{\prime\prime}{7} = 1$ *Periodical Month*, or the Time which the Moon takes up in finishing her Course round the Earth.

$\overset{D.}{29} + \overset{H.}{12} + \overset{\prime}{44} + \overset{\prime\prime}{08} = 1$ *Mean Synodical Month* or *Lunation*, i. e. the mean Time between Conjunction and Conjunction.

The *Calendar* or *Civil Month* consists of sometimes 30 *Days* sometimes 31; and one *Month*, viz. *February*, of 28 or in *Leap-Year* 29 *Days*.

$\overset{D.}{365} + \overset{H.}{5} + \overset{\prime}{48} + \overset{\prime\prime}{57} = 1$ *Solar* or *Tropical Year*, i. e. the Time which the Sun takes up in finishing its apparent Course thro' the 12 *Signs* of the *Zodiack*.

$\overset{D.}{365} + \overset{H.}{6} + \overset{\prime}{9} + \overset{\prime\prime}{14} = 1$ *Syderial*, *Anomalistical*, or *Periodical Year*; the Space of Time which the Sun takes up in finishing its apparent Course round the Earth.

The *Civil*, *Julian*, or *Calendar Year*, is the Space of 12 *Calendar Months*, or of 365 *Days* 6 *Hours*; or it is the Space of 365 *Days* every *Common Year*, and of 366 every *Fourth Year*, which is therefore called *Bissextile* or *Leap-Year*, because of the *Day* added.

DEFINITION XXVI.

308. The *Golden Number*, *Cycle of the Moon*, or *Metonic Cycle* (so called from itis Author *Meton*) is a Period of 19 *Julian Years*, which is supposed to be

be equal to 235 *Lunations*, ſo that every new and full Moon is computed to return to the ſame Day of the Month it was 19 Years before, but with the Errour of about $1\frac{1}{2}$ Hour in Point of Defect. The Beginning of this *Cycle* was the Year before *Chriſt*.

Definition XXVII.

309. The *Cycle of the Sun* is the Space of 28 Years, in which Time all the Days of the Year return to the ſame Day of the Week that they were 28 Years before. The Beginning of this *Cycle* was 9 Years before *Chriſt*.

Chap. IV.

Of the Addition and Subtraction of Polynomial Numbers.

Partition VIII.

310. ADDITION and *Subtraction* of Polynomial Numbers is either *Pure* or *Mixed*, according as the Numbers to be added or ſubtracted are ſuch.

Problem I.

311. To add two or more given Homogeneous Polynomes into one Sum.

Effection.

Pre. 1. Set all the Homogeneous Figures which belong to each given Number in the ſame Column one under another (In. 67.)

Pre. 2. Draw a Line beneath all.

Pre. 3. Begin at the loweſt Figure in the firſt Column to the Right-hand, and add up all the Figures in that Column marking all the Tens.

Pre. 4. Set down the Remainder above all the Tens in the firſt Column.

Pre. 5. Add the Number of Tens in ſumming up the firſt Column to the loweſt Figure in the ſecond Column, and proceed to ſum up that Column in the ſame manner as the firſt.

Pre. 6. Thus continue doing thro' all the Columns.

Pre. 7. And becauſe in ſumming up the laſt Column, there remains no other Column for the Tens to be added to; therefore they are to be ſet down by themſelves: And the Number thus found will be the Sum required (In. 35.) Q. E. E.

Ex. gr.

Ex. gr. Let it be required to find the Sum of the given Numbers 4037.654+7956.508+684.097+5932.630. First set down the Numbers as in the Margin, then, beginning at the lowest Figure in the first Column to the Right hand, which is here the Fractional Place of Thousandths, say 7 and 8 is 15, and 4 is 19, or $7+8+4=19=10+9$, set down 9 and carry 1 for the 10; $1+3+9+5=18$, set down 8 and carry 1; $1+6+5+6=18$, set down 8 and carry 1; $1+2+4+6+7=20$, set down 0 and carry 2; $2+3+8+5+3=21$, set down 1 and carry 2; $2+9+6+9=26$, set down 6 and carry 2; $2+5+7+4=18$, which set down by Precept 7. Therefore the Sum required is 18610.889. *i. e.* 18 Thousand, 6 Hundred, and Ten, with 889 Thousandth Parts.

$$\begin{array}{r}
4037.654 = b \\
7956.508 = c \\
0684.097 = d \\
\underline{5932.630 = f} \\
18610.889 = b+c+d+f.
\end{array}$$

Problem II.

312. To subtract a lesser given *Pure Polynome* from a greater given one.

Effection.

1. Place all the Homogeneous Figures in the same Column one beneath another (In. 67.)
2. Draw a Line beneath all.
3. Begin at the lower Figure in the first Column to the Right-hand, and subtract every lower Figure in each Column from its respective one above.
4. When the upper Figure is lesser add 10 to it, in such Case always observing either to take an Unit from the next upper Figure to the Left-hand, or which is the same Thing add an Unit to the next lower Figure (In. 68.)

And the Number thus found will be the Difference required. Q. E. E.

Ex. gr. Let it be required to take the Number 730.25 from the Number 940.18. First set down the Numbers as in the Margin, then, beginning at the lower Figure in the first Column to the Right-hand, say 5 from 8 leaves 3 to be set down under the first Column; 2 from 1 I cannot take, but 2 from $10+1=11$ leaves 9 to be set under the second Column; then for the 10 that I added to the upper Figure in the second Column I add 1 to the lower Figure in the third Column, thus $0+1=1$, and proceed saying 1 from 0 I cannot, but 1 from $10+0=10$ leaves 9, which I set under the third Column; then 1 that I add to 3 in the fourth Column for the 10 added to 0 in the third makes 4, 4 from 4 leaves 0, which I set under the fourth Column: Lastly, 7 from 9 leaves 2, which I set down under the last Column. And the Remainder is found to be 209.93.

Subtrahend	940.18	$= b$
Minuend	730.25	$= c$
Remainder	209.93	$= b - c$

Problem III.

313. To add together two or more *Mixed Polynomes* into one Sum.

Effection.

Effection.

The Effection of this Problem is directly the ſame with that of Problem 1. of this Part, only, inſtead of marking (or pointing) all the Tens, as is there directed, take Notice here how many of each ſuperiour Denomination in the given Polynomes are contained in the Sum of all the Figures of the next inferiour Denomination, and ſo many Units are to be added to the Figures of that ſuperiour Denomination.

Ex. gr. Let it be required to add together the ſeveral Sums of Money ſet down in the Margin.

l.	*s.*	*d.*	*qr.*
256	13	07	01
79	09	10	03
63	17	11	00
42	00	03	02
442	01	08	02

Here beginning as before at the loweſt Figure to the Right-hand, ſay 2+3+1=6=1*d.*+2*qr.* (In. 290) ſet down the 2*qr.* under their proper Column, and carry the 1*d.* to the next; 1+3+11+10+7=32*d.*=2*s.*+8*d.* ſet down 8 and carry 2*s.*; 2+17+9+13=41*s.*=2*l.* +1*s.* ſet down 1 and carry 2; 2+2+3+9+6=22, ſet down 2 and carry 2; 2+4+6+7+5=24, ſet down 4 and carry 2; 2+2=4. Therefore the Sum required is 442*l.* 01*s.* 08*d.* 02*qr.*

Ex. 2. Sold at ſeveral Times to Mr. *Thomas Traffick*, 1732, *Jan.* 13.

	l.	*s.*	*d.*
—— Yards of —— at —— *per* Yard	42	13	00
Feb. 4th —— C of Flax at —— *per.* C.	79	11	00
April 29th —— Ells of Shalloon at —— *per* Ell	170	15	06
In all	292	19	06

Examples in Addition of Weights.

Troy Weight.

	lb.	*oz.*	*p.w.*	*gr.*
Bought Amber at ſeveral times	15	05	00	21
	03	09	15	00
	10	11	12	15
	07	10	19	20
Sum	38	01	08	08

Averdupois Weight.

	C.	*Q.*	*lb.*	*Oz.*
Bought at ſeveral times Tobacco	10	02	20	11
	14	03	24	15
	18	01	27	11
	19	03	14	15
Sum	64	00	04	04

Addition

Addition of Long Measure.

	Yds.	Qrs.	Nails.
	23	01	02
	19	03	01
	14	00	03
	06	00	02
	04	02	03
		01	00
Sum	68	01	03

	Yds.	Feet.	Inches.
	70	02	09
	26	01	11
	43	01	10
	27	00	09
	20	01	04
	17	02	11
Sum	206	02	06

	Leagues.	Miles.	Furlongs.
	24	02	07
	23	01	06
	20	01	01
	15	02	06
	09	00	07
Sum	94	01	04

And so for any other Kind of Weight or Measure.

PROBLEM IV.

314. To subtract a lesser given *Mixed Polynome* from a greater.

Effection.

The Effection of this Problem is in all respects the same with Subtraction of Pure Polynomes, only when the upper Figure is less, instead of adding 10 to it, as is there directed, let it be encreased by the Addition of as many of its own Denomination as make one of the next superiour Denomination.

Ex. gr. Suppose it be required to subtract the lower Sum of Money in the Margin from the upper one.

l.	*s.*	*d.*	*qr.*
432	11	05	01
252	11	07	02
179	19	09	03

Here beginning as before, say, 2 from 1 cannot, but 2 from 4 + 1 = 5*qr.* leaves 3 *qr.* then for the 4 *qr.* = 1*d.* which I added to the upper Figure in the Farthings Column, I add 1*d.* to the lower Figure in the Pence Column, saying, 1+7 =8; 8 from 5 I cannot but 8 from 12+5=17 leaves 9*d*; then, for the 12*d* = 1*s.* I added to the upper Figure in the Pence Column, add 1*s.* to the lower Figure in Shillings Column, saying 1+11=12; 12 from 11 I cannot, but 12 from 20+11=31 leaves 19*s.* then for the 20*s.*=1*l.* added to the upper Figure in Shillings Column I add 1*l.* to the first lower Figure in the Pounds Column, saying 1+2=3, 3 from 10+2=12 leaves 9; 1+5=6, 6 from 10+3=13 leaves 7; 1+2=3, 3 from 4 leaves 1; therefore the Remainder required is 179*l.* 19*s.* 09*d.* 03*qr.*

Ex. 2. Suppose a Writing drawn *April* 3d, 1680. I demand how long it is since, this present 18th of *August* 1734?

Yrs.	*M.*	*W.*	*D.*
1734	08	00	06
1679	03	01	01
55	04	03	05

Observe from your Calendar that from *New Year's Day* 1680 to *April* 3d is just 13 Weeks 1 Day, it being *Leap-Year*: and from *New Year's Day* 1734 to *August* 18 is 32 Weeks 6 Days. Therefore accounting 4 Weeks to the Month, the Age of the Writing will be found to be, as in the Margin.

Examples

Examples in Subtraction of Weights.

Troy Weight.

	lb.	oz.	p.w.	gr.
From	13	07	03	05
Take	12	07	15	05
Remains	00	11	08	00

Averdupois Weight.

	Tun.	C.	Q.	lb.	Oz.	Dr.
From	5	15	01	23	12	03
Take	3	16	01	21	15	02
Remains	1	19	00	01	13	01

PROBLEM V.

315. To prove Addition, *i. e.* to know whether the Sum found be the true Sum or no.

Effection.

1. Sum up every Column by it felf beginning with the uppermoft Figure to the Left-hand.
2. Take the Sum of all thefe, and if that equals the Sum found, the Addition is right, otherwife not (In. 23.)

Ex. gr. Let it be required to prove whether the Sums found in the Examples (In. 311 and 313) be the true Sums or not.

	abcdefg
	4037.654
	7956.508
	0684.097
	5932.630
	18610.889
a =	16
b =	24
c =	19
d =	19.
e =	1.7
f =	.17
g =	19
Proof	18610.889

l.	*s.*	*d.*	*qrs.*
256	13	07	01
79	09	10	03
63	17	11	00
42	00	03	02
442	01	08	02

Sum of	Pounds Column =	2			
		22			
		20			
	Shillings Column =	1	19		
	Pence Column =		02	07	
	Farthings Column =			01	02
Proof	— —	442	01	08	02

PROBLEM VI.

316. To prove Subtraction, *i. e.* to know whether rhe Remainder found be the true Remainder fought or not.

Effection.

Add together the Subtrahend and the Remainder, and if the Sum be the fame with the Minuend, the Work is right, otherwife not (In. 68.) Q. E. E.

Let

Let it be required to prove whether the Remainders found in the Examples (In. 312 and 314) be the true Remainders required.

The Work is as follows.

				l.	*s.*	*d.*	*qr.*	
Minuend	940.18	$=b$	*Minuend*	432	11	05	01	$=b$
Subtrabend	730.25	$=c$	*Subtrabend*	252	11	07	02	$=c$
Remainder	209.93	$=b-c$	*Remainder*	179	19	09	03	$=b-c$
Proof	940.18	$=b-c+c=b$	*Proof*	432	11	05	01	$=b-c+c=b$

Chap. V.

Of Multiplication and Division.

Problem VII.

317. To make the *Multiplication Table*, or to find the Product to all the 9 Digits.

1. Write down the 9 Digits
2. Add 2+2, 3+3, 4+4, 5+5, &c.
3. To the last Sums under 3, 4, 5, &c. add 3, 4, 5
4. To the last Sums under 4, 5, 6, &c. add 4, 5, 6, &c.
5. To the last Sums under 5, 6, 7, &c. add 5, 6, 7, &c.
6. To the last Sums under 6, 7, 8, &c. add 6, 7, 8, &c.
7. To the last Sums under 7, 8, 9; add 7, 8, 9.
8. To the last Sums under 8, 9; add 8, 9.
9. To the last Sum under 9 add 9; and it's done.

1	2	3	4	5	6	7	8	9
	4	6	8	10	12	14	16	18
		9	12	15	18	21	24	27
			16	20	24	28	32	36
				25	30	35	40	45
					36	42	48	54
						49	56	63
							64	72
								81

Scholium III.

318. It is necessary that this Table be got perfectly by Heart, before the Learner proceed to Multiplication and Division. Its first Inventor is said to have been *Pythagoras* of *Samos*, whence it is stiled the *Pythagoric Abacus*.

Problem VIII.

319. To Multiply by a *Monome*.

Effection.

1. Set the Multiplier under that Figure of the Multiplicand, which is of the ſame Denomination, and draw a Line under both.
2. Begin with the loweſt Figure in the Multiplicand, and proceed in ſuch Sort, that the Tens which are gained by multiplying each lower Figure be added to the Product of the next higher.
3. Count as many Cyphers or Fractional Places to the Right-hand in the Product, as are in both Factors, and it is done. Q. E. F.

Ex. gr. Let it be required to multiply 6780.5 by 0.4. Set down the given Factors as in the Margin, then ſay $4 \times 5 = 20$, ſet down 0 and carry 2; $4 \times 0 = 0$ and 2 that was carried makes 2, ſet down 2 and carry 0; $4 \times 8 = 32$, ſet down 2 and carry 3; $4 \times 7 = 28$, and 3 that was carried makes 31, ſet down 1 and carry 3; $4 \times 6 = 24$ and 3 that was carried makes 27; Therefore the Product required is 2712.20 or 2712.2 (In. 268.) Q. E. E.

$$\begin{array}{r} 6780.5 \\ 0.4 \\ \hline 2712.20 \end{array}$$

Ex. 2. Let it be required to multiply 72000 by 900. The Work is in the Margin.

$$\begin{array}{r} 72000 \\ 900 \\ \hline 64800000 \end{array}$$

Problem IX.

320. To Multiply by a *Polynome.*

Effection.

1. Set each Figure in the Multiplier under its Homogeneous one in the Multiplicand, and draw a Line beneath them, as in the laſt.
2. Multiply by each Figure in the Multiplier ſingly as in the laſt.
3. Obſerve ſuch Order in ſetting the particular Products beneath one another, that the loweſt Figure in each be ſtill ſet under its reſpective multiplying Figure.
4. Add all the particular Products as they ſtand, into one Sum.
5. Annex the Cyphers to that Sum, which are in both Factors, if any there be: or cut off from it as many Fractional Places as are in both Factors (In. 283) and it is done. Q. E. E.

Ex. gr. Let it be required to multiply 5432.01 by 960.32. The Operation follows.

$$\begin{array}{rcl} 5432.01 & & \textit{Multiplicand.} \\ 960.32 & & \textit{Multiplier} \\ \hline 1086402 & = & 000.02 \\ 1629603\,. & = & 000.30 \\ 3259206\ldots & = & 060.00 \\ 4888809\ldots. & = & 900.00 \\ \hline 5216467.8432 & = & 960.32 \times 5432.01 \text{ the } \textit{Product.} \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \times 5432.01$$

Problem X.

Problem X.

321. To multiply large Numbers without the help of a *Multiplication Table*. *Ex. gr.* Suppose it were required to multiply 507.8420063901 = *A* by 87.0600914532 = *B*,

Effection.

1. Set down the 9 Digits one under another as below.
2. Set down the Multiplicand against 1, for 1 × *A*: for 2 × *A* set down its Double: for 3 × *A* add 1 and 2 *A*: for 4 × *A* add 3 *A* and 1 *A*: for 5 *A* add 2 *A* and 3 *A*: for 6 *A* take twice 3 *A*: for 7 *A* add 3 *A* and 4 *A*: for 8 *A* take twice 4 *A*: for 9 *A* add 4 *A* and 5 *A*: and thus you will have a *Table* or *Tariffa* of the Multiplicand to every Figure in the Multiplier, as below.
3. Set down the given Factors *A*, *B*, with a Line drawn beneath them, as in the Left.
4. Insert the particular Products out of the *Tariffa* one beneath another, as in the Left. And the Sum of these (cutting off as many Decimals as are in both Factors) will be the Product required.

Tariffa.

1		= 5078420063901	=	1 *A*
2		= 10156840127802	=	2 *A*
3		= 15235260191703	=	3 *A*
4		= 20313680255604	=	4 *A*
5	× *A*	= 25392100319505	=	5 *A*
6		= 30470520383406	=	6 *A*
7		= 35548940447307	=	7 *A*
8		= 40627360511208	=	8 *A*
9		= 45705780575109	=	9 *A*

	507.8420063901 = *A*.
	87.0600914532 = *B*.
	10156840127802
1	5235260191703
25	392100319505
203	13680255604
507	8420063901
45705	780575109
30470520	383406
3554894044	7307
40627360511	208
44212.771520	09868477509332 = *A* × *B*.

But,

But becaufe it is generally fufficient, if we can only have fix or feven Decimal Places true in the Product; therefore the following Problem is of great Ufe to contract the Work in large Multiplications.

PROBLEM XI.

322. To abbreviate large Multiplications fo as to retain no more than an affigned Number of Fractional Places in the Product.

Effection.

1. See how many Places of Integers there will be in the Product from the given Factors (In. 281) and adding to that Number the affigned Number of Fractional Places, note the Sum.
2. Retain as many Figures in each Factor as that Sum, and one Figure more: and if either Factor want of fuch Number of Places, fupply it with Cyphers in the Places of Fractions.
3. Make a Tariffa of the Multiplicand to every Figure in the Multiplier.
4. Set down the Multiplicand in its direct Order, and the Multiplier under it in an inverted Order: *i. e.* fet the higheft Figure of the Multiplier under the loweft Figure of the Multiplicand, and the next higheft Figure of the Multiplier under the next loweft of the Multiplicand, *&c.*
5. Set down the particular Products one under another, in fuch Sort, that each Product reach to no more Places in the Multiplicand than what ftands to the Left-hand over its Multiplying Figure: obferving to place the loweft Figure of each in a direct Line beneath one another, and withal retaining the Increafes of all the loweft Figures, as in the Tariffa.
6. Add up the particular Products, as they ftand, and the Sum will be the Product required, true in all it's Places, except the one or two loweft. Q. E. E.

Ex. gr. Let it be required to multiply 507.84200063901 by 87.0600914532 fo as to retain 6 Fractional Places true in the Product.

The Places of Integers in the Product will be 5 (In. 281) which with the fix Places of Fractions will make up in all eleven Places: Therefore by Pre. 2. the Factors are to confift of 12 Places of Figures each. The whole Operation will ftand as follows.

The

The *Tariffa.*

1	50784200639
2	101568401278
3	152352601917
4	203136802556
5	253921003195
6	304705203834
7	355489404473
8	406273605112
9	457057805751

507.842006390		
2354190060.78		the *Multiplier* inverted.
40627.36051120	=	80. × 507.842006390
3554.89404473	=	7. × 507 84200639 + 0
30.47052038	=	0. 06 × 507842006 + 2
4570578	=	0. 00009 × 507842 + 0
50784	=	0. 000001 × 50784 + 0
20313	=	0. 0000004 × 5078 + 1
2539	=	0. 00000005 × 507 + 4
152	=	0. 000000003 × 50 + 2
10	=	0. 0000000002 × 5 + 0

The *Product* = 44212.77152007 only two little in the lowest Figure by 2.

The Reason of this Operation will appear plainly to any one who considers that all the particular Products here, are the same with those in the other Method, only rejecting a certain Number of inferiour Places, and consequently the whole must be the same too (In. 23.)

Problem XII.

323. To divide by a *Monome.*

Effection.

1. Set down the Divisor to the Left-hand of the Dividend with a Curve Line between them.
2. Make another Curve Line to the Right-hand of the Dividend, for the Place of the Quotient.
3. Ask how often the Divisor can be contained in the highest, or (if not) in the two highest Figures of the Dividend, which call the *First Dividual.*
4. Set the Answer behind the Curve Line to the Right-hand for the first Quotient Figure, and note its Place (In. 286.)
5. Multiply the first Quotient Figure into the Divisor, and subtract the Product from the *First Dividual.*
6. To the Remainder draw down another Figure or Cypher out of the Dividend, for a *Second Dividual,* making a Point in the Dividend under every Figure so drawn down.
7. Ask how often the Divisor can be contained in the *Second Dividual,* and if it cannot be contained in it, draw down Figures or Cyphers out of the Dividend, till it can, observing to make a Cypher in the Quotient for every Figure or Cypher drawn to the Remainder, besides the first; and

continue to make Points under every Figure in the Dividend, as they are drawn down.

8. Set the Answer again to the Right-hand of the first Quotient Figure, or to the Right-hand of the Cypher or Cyphers, if any there be, which were set in the Quotient by the last Precept.

9. Multiply the second Quotient Figure into the Divisor and subtract the Product from the *Second Dividual*, and to the Remainder draw down another Figure or Cypher out of the Quotient for a *Third Dividual.*

10. Ask how often the Divisor can be contained in the *Third Dividual*, and if it cannot be contained in it draw down Figures out of the Dividend, as before, till it can, still observing to make a Cypher in the Quotient for every Figure or Cypher drawn to the Remainder, besides the first.

11. Thus continue doing as long as any thing remains, or if something always remain, carry on the Operation to what Number of Fractional Places you please. Q. E. E.

Ex. 1. Let it be required to divide 2712.26 by 0.4. Here the *First Dividual* is 2700, but $\frac{2700}{0.4}$ is $>$ 6000 (In. 283) or the 4s in 27 are 6, therefore 6 (or 6000) is the first Quotient Figure. The Operation is as follows.

Divisor, Dividend, Quotient.
0.4) 2712.26 (6780.65

6000 × 0.4 = 24 . . *subtract.*

Second Dividual 31
700 × 0.4 = 28 *subtract.*

Third Dividual 32
80 × 0.4 = 32 *subtract.*

Fourth Dividual = 026
0.6 × 0.4 = 24 *subtract.*

Fifth Dividual = 20
0.05 × 0.4 = 20 *subtract.*

Remains (0)

Ex. 2. Let it be required to divide 62.85 by 0.007. Here the first Quotient Figure must be in the Place of Thousands, i. e. $\frac{0062.85}{0.007}$ $>$ 8000 (In. [illegible].)

.007)

.007) 62.85 (8978.571428571428571428 &c. *ad infinitum.*

```
 56
 ---
  68
  63
  ---
   55
   49
   ---
    60
    56
    ---
     40
     35
     ---
      50
      49
      ---
       10
        7
       ---
        30
        28
        ---
         20
         14
         ---
          60
          56
          ---
          40 &c.
```

In this Operation you see that the same Figures return at the sixth Place below Unity.

SCHOLIUM IV.

324. But the Division by one Figure is more compendiously performed by *Halfing*, *Thirding*, *Fourthing*, &c. the Dividend.

Ex. gr. Suppose it were required to take the seventh Part of 22407, say the $\frac{1}{7}$ of 22 is 3, and the Remainder is 1, to which annex the 4; and say the $\frac{1}{7}$ of 14 is 2; the $\frac{1}{7}$ of 0 is 0; the $\frac{1}{7}$ of 7 is 1: Therefore the Quotient of 22407 divided by 7 is 3201, as in the Margin. Thus again in taking the eighth Part of 4165856, say the 8th of 41 is 5 and there remains 1, to which annex 6; the 8th of 16 is 2; the 8th of 5 I cannot, but the 8th of 58 is 7, and the Remainder is 2; the 8th of 25 is 3, and the Remainder is 1; the 8th of 16 is 2; Therefore the Quotient is 520732.

$\frac{1}{7}$ | 22407 / 3201

$\frac{1}{8}$ | 4165856 / 520732

SCHOLIUM

Scholium V.

325. The *Fifthing* of any Number is best performed by doubling it, and then reducing all the Places in the Product one Place lower, or by multiplying it by 0.2. *Ex. gr.* Suppose it were required to take $\frac{1}{5}$ Part of 2037.5, the double of 2037.5 is 4075.0: therefore the $\frac{1}{5}$ Part is 407.5, and so for any other.

$$\begin{array}{r|r} \frac{1}{5} & 2037.5 \\ & 0.2 \\ \hline & 407.5 \end{array}$$

Problem XIII.

326. To divide by a given *Polynome.*

Effection.

1. Set down the given Divisor and Dividend, as in the last Problem.
2. From the highest Figures in the Dividend assume the same Number of Places that are in the Divisor of the *First Dividual*; or if the Figures in the Divisor exceed the same Number of Places in the Dividend, assume one Place more out of the Dividend, for the *First Dividual.* Then proceed directly, as in the last Problem: Only the Quotient Figures here are determined by comparing the highest Figures alone of each *Dividual* with the highest Figures of the Divisor, omitting the rest, as will be best seen by an Example.

Example 1.

Let it be required to divide 5216467.8432 by 960.32. Here the Numbers being set down as is directed, the first Dividual will be 521646 (or 5216460) whence the first Quotient Figure of $\frac{5216460}{960.32}$ or which is nearly the same $\frac{5200000}{900.}$ or $\frac{52000}{9}$ will be in Thousands Place; *i. e.* the 9*s* in 52000 are the same with the 960.32*s* in 5216460, which are 5000 for the first Quotient Figure; and $5 \times 96032 = 480160$, $521646 - 480160 = 41486$, to which draw down 7 for a second Dividual 414867; then the 96032*s* in 414867, or which is the same the 9*s* in 41 are 4 for the second Quotient Figure; $4 \times 96032 = 384128$, $414867 - 384128 = 30739$, to which draw down 8 for a third Dividual 307398; then the 96032*s* in 307398, or the 9*s* in 30 are 3, for the third Quotient Figure, $3 \times 96032 = 288096$, $307398 - 288096 = 19302$, to which draw down 4 for a fourth Dividual; the 96032*s* in 193024 or the 9*s* in 19, are 2 for the fourth Quotient Figure, $2 \times 96032 = 192064$, $193024 - 192064 = 960$, to which bring down 3, and because the Divisor 960.32 cannot be con-

```
960.32)5216467.8432(5432.01
       480160
       ------
        414867
        384128
        ------
         307398
         288096
         ------
          193024
          192064
          ------
            96032
            96032
            -----
             (0)
```

contained in the Dividual 9603, therefore the Figure 2 muſt be alſo drawn down, obſerving to place a Cypher in the Quotient; laſtly the 96032s in 96032, or the 9s in 9 are 1 for the fifth Quotient, which finiſhes the Diviſion without a Remainder, as in the foregoing Margin.

Example 2.

Let it be required to divide 694943164 by 57.91. Here the firſt Dividual is 6949, and the Quotient Figure of $\frac{694943164}{57.91}$ is in the 10000000s Place (In. 286) then becauſe the ſecond Figure in the Diviſor is greater in its Place than the firſt, therefore $\frac{69\,40}{57.91}$ is nearly the ſame with $\frac{69}{57}$ *i. e.* the 57s in 69 are the ſame with the 5791s in 6949, which gives 1 (or 10000000) for the firſt Quotient Figure,

```
57.91)694943164(12000400
            .....
      5791
      11584
      11582
        23164
        23164
         (o)
```

$1 \times 57.91 = 57.91, 6949 - 5791 = 1158$, to which draw down 4 for a ſecond Dividual; then the 5791s in 11584 are 2 for the ſecond Quotient Figure, $2 \times 5791 = 11582$, $11584 - 11582 = 2$, to which draw down 3 for a third Dividual 23, but becauſe the given Diviſor cannot be contained in this Dividual without drawing to it the other three Figures 164 out of the Dividend, and ſo making it inſtead of 23 to be 23164, therefore 3 Cyphers are to be ſet in the Quotient before the third Quotient Figure, which is 4, or 400 (In. 286) and the Diviſion is finiſhed without a Remainder, as in the Margin.

SCHOLIUM VI.

327. When the Diviſor is an Integer with Cyphers to the Right-hand, if thoſe Cyphers be cut off, and the Dividend depreſſed ſo many Places lower, the Quotient will be ſtill the ſame (In. 286) as in the following Examples.

```
72|000)64800|000(900        12|0)172|8(14.4
       648
```

SCHOLIUM VII.

328. When the Diviſor is a Fraction with Cyphers to the Left-hand, thoſe Cyphers may be cut off, and the Dividend advanced ſo many Places higher, and the Quotient will be ſtill the ſame, as in the following Examples.

$$\frac{0.0648}{0.0072} = \frac{6.48}{0.72} = 9 : \frac{0.0648}{0.0072} = \frac{64800}{0.72} = \frac{6480000}{72} = 900$$

PROBLEM XIV.

329. To abbreviate large Divisions, so as to retain no more than an assigned Number of Fractional Places in the Quotient.

Effection.

1. See the Place of the first Figure in the Quotient, by comparing the Dividend and Divisor (In. 286.)
2. Retain as many of the Figures of the highest Places in the Dividend, as you design there should be in the Quotient, and one Figure or Cypher more.
3. Retain the same Number of the Figures of the highest Places in the Divisor, if the highest Figure in the Divisor be less than the highest in the Dividend, if not, retain one Figure fewer.
4. Make a Table or Tariffa of the Divisor to all the 9 Digits.
5. Set down the Dividend and Divisor in their direct Order as (In. 326.)
6. Subtract the greatest Product you can find in your Tariffa beneath the Dividend, setting down it's multiplying Figure for the first Figure in the Quotient; and the Remainder without annexing any other Figures to it is the second Dividual.
7. Subtract the greatest Product you can from that second Dividual, rejecting as many of its lowest Figures, as it exceeds the Number of the Places of it's Dividual, and set down the multiplying Figure for the second Figure in the Quotient, and the Remainder without annexing any other Figures to it will be the third Dividual.
8. Subtract the greatest Product you can from that third Dividual, proceeding just as before.
9. Continue so doing as long as any thing remains to divide by, observing that for every Place above one your Divisor is shortned for any Dividual, a Cypher must be inserted in the Quotient. And the Quotient thus found is in general true in all its Places, except perhaps the lowest.

Ex. gr. Let it be required to divide 44212.771520098684775 *&c.* by 507.8420063901 so as to retain seven Places of Fractions true in the Quotient.

The Place of the highest Figure is that of Tens (In. 286) which with the seven Places of Fractions make in all nine. Therefore by Pre. 2. the Dividend is to consist of ten Places, and the Divisor of nine. The whole Operation is as follows.

The

The *Tariffa.*

1	507842006
2	1015684012
3	1523526019
4	2031368025
5	2539210031
6	3047052038
7	3554894044
8	4062736051
9	4570578057

)44212.77152(87.06009145 the Quotient.
4062736051 = 80 × 507.842006
358541100
355489404 = 07 × 507.84200 + 4
3051696
3047052 = 00.06 × 507.842 + 6
4644
4570 = 00.00009 × 507 + 7
73
50 = 00.000001 × 50 + 0
23
20 = 00.0000004 × 5 + 3
2 &c.

Problem XV.

330. To prove *Multiplication.*

Effection.

Divide the Product by either Factor, and if the Quotient be equal to the other Factor the Work is right, otherwise not (In. 83.) Q. E. E.

Problem XVI.

331. To prove *Division.*

Effection.

Multiply the Quotient into the Divisor, and if the Product be equal to the Dividend the Work is right, otherwise not (In. 83.) Q. E. E.

Problem XVII.

332. To find the greatest *Common Measure* between any two given Numbers (In. 135.)

Ex. gr. Let it be required to find the greatest *Common Measure* between 4389 and 3927. The Operation is as follows.

3927)4389(1
3927
462)3927(8
3696
231)462(2
462
...

Therefore the greatest *Common Measure* is 23.

C H A P.

CHAP. VI.

Of the Reduction of Coin, Weights, Measures, *&c.*

DEFINITION XXVIII.

333. BY *Reduction* is here meant the bringing Numbers of one Denomination into Numbers of another Denomination.

PARTITION IX.

334. *Reduction* is stiled *Descending* or *Ascending*, the former is performed by Multiplication, the latter by Division.

DEFINITION XXIX.

335. That is termed *Reduction Descending*, which brings Numbers of a higher into a lower Denomination by Multiplication, as of Pounds into Shillings, of Shillings into Pence, *&c.* in Coin; of Pounds into Ounces, of Ounces into Penny Weights, *&c.* in Troy Weight; *&c.*

PROBLEM XVIII.

336. To reduce a Number of a higher into a lower Denomination.

Effection.

Consider how many Units of that lower Denomination are contained in an Unit of the given Number (according to the Table of Coin, Weight, Measure, *&c.* to which it belongs) and the Product which arises by the Multiplication of the given Number into that Tabular Number is the Number required. Q. E. E.

Example.

Let it be required to reduce 369*l.* *Sterling* into Shillings, those Shillings into Pence, and those Pence into Farthings.

	l.	
	369	
Multiply by	20	the Shillings in 1*l.* *Sterling* (Th. 292.)
	7380	= the Shillings in 369 *l.*
Multiply by	12	the Pence in one Shilling.
	14760	
	7380	
	88560	= the Pence in 369*l.*
Multiply by	4	the Farthings in one Penny.
	354240	= the Farthings in 369*l.*

Other-

Otherwise the given Number may be reduced into Pence or Farthings at one Operation, thus,

```
                  369                                    l.
Multiply by  240 the Pence in one Pound.                369
             ----                                      ×980 the Farthings in 1l.
             1476                                      ----
             738                                       2214
             -----                                    3321
             88560=the Pence in 369l. as before.     ------
                                                     354240 = the Farth. in 369l.
```

Example 2.

```
        l.     s.    d.    qrs.
Reduce 835 . 13 . 11 . 03 into Farthings (In. 292.)
       ×20
      -----
      16700 = Shillings in 835l.
        +13 the odd Shillings add
      -----
      16713
        ×12
      -----
      33426
     16713
     ------
     200556 = Pence in 835l. 13s.
        +11 the odd Pence add
     ------
     200567 = the Pence in 835l. 13s. 11d.
         ×4
     ------
     802268 = the Farthings in 835l. 13s. 11d.
         +3 the odd Farthings add
     ------
     802271 = the Farthings in 835l. 13s. 11d. 03qr. as was required.
```

Otherwise this Work may be shortned by taking in the odd Shillings when you multiply by 20, the odd Pence when you multiply by 12, the odd Farthings when you multiply by 4; and the same may be understood for other Kinds of Reduction, as follows.

```
                     835l. 13s. 11d. 03qr.
Multiply by             20 and take in the odd 13s.
                    ------
                    16713
Multiply by            12  and take in the odd 11d.
                   ------
                   200567
Multilpy by             4  and take in the odd 3qrs.
                   ------
                   802271  the Farthings in 835l. 13s. 11d. 03qrs. as above.
```


Reduce

Example 3.

Reduce 34*lb.* 7*oz.* 13*p.w.* 21*gr.* Troy Weight into *Grains* (In. 293.)

$$
\begin{array}{rl}
 & \times 12 \\
\hline
415 & = \text{the } \textit{Ounces} \text{ in } 34\textit{lb.}\ 7\textit{oz.} \\
\times 20 & \\
\hline
8313 & = \text{the } \textit{Penny Weights} \text{ in } 34\textit{lb.}\ 7\textit{oz.}\ 13\textit{p.w.} \\
\times 24 & \\
\hline
33253 & \\
16628 & \\
\hline
199533 & = \text{the } \textit{Grains} \text{ in } 34\textit{lb.}\ 7\textit{oz.}\ 13\textit{p.w.}\ 21\textit{gr.} \text{ as required.}
\end{array}
$$

The following Examples with their Anſwers are inſerted for the Learner's Practice.

Ex. 4. In 5*C.* 01*qr.* 23*lb.* 13*oz.* *Averdupois* how many *Ounces* ?
Anſwer, 9789 *Ounces*.

Ex. 5. In 360 Degrees (In. 297, 298) How many Barley Corns ?
Anſwer, 4752000000 Barley Corns.

Ex. 6. In 1734 *Julian* Years, How many Seconds (In. 307.) ?
Anſwer, 54720878400 Seconds.

Scholium VIII.

337. Whenever you are to multiply by 12, the Work may for Brevity's Sake be performed (as in the ſecond and third Examples above) by help of the following Table.

$$
12 \times \left\{ \begin{array}{l} 1 = 12 \\ 2 = 24 \\ 3 = 36 \end{array} \right\}
\quad 12 \times \left\{ \begin{array}{l} 4 = 48 \\ 5 = 60 \\ 6 = 72 \end{array} \right\}
\quad 12 \times \left\{ \begin{array}{l} 7 = 84 \\ 8 = 96 \\ 9 = 108 \end{array} \right\}
$$

Definition XXX.

338. *Reduction Aſcending* is the Converſe of *Reduction Deſcending*, or it is the bringing Numbers out of a lower into a higher Denomination by Diviſion ; as of Farthings into Pence, of Pence into Shillings, and Shillings into Pounds.

Problem XIX.

339. To reduce a given Number out of a lower into a higher Denomination.

Effection.

Conſider how many Units of the given Number are contained in an Unit of the next higher Denomination, according to the Table of Coin, Weight,

Meaſure,

Meaſure, &c. to which it belongs; and the Quotient reſulting from the Diviſion of the given Number by that Tabular Number is the Number required. And if any thing remain after the Diviſion is finiſhed, it is of the ſame Denomination with the Dividend. Q. E. E.

Example 1.

Let it be required to determine how many Pence Shillings and Pounds are contained in 802271 Farthings?

```
            12)      2|0)
4)802271(200567(1671|3(835l.
   22      80       7
   27      85       11
   31      16       13s.
   3qr.    47
           11d.
```

Anſwer, 855*l*. 13*s*. 11*d*. 03*qrs*.

Ex. 2. In 199533 *Grains*, How many *Penny Weights*, *Ounces*, and *Pounds Troy*?

```
           2|0)   12)
24)199533(831|3(415(34lb.
   192       3    55
    75      11    7oz.
    72     13p.w.
    33
    24
    93
    72
    21gr.
```

Anſwer, 34*lb*. 7*oz*. 13*p.w.* 21*gr*.

Ex. 3. In 9787 *Ounces*, How many *Pounds*, *Quarters*, and *Hundreds Averdupois*?

Anſwer, 5*C*. 01*qr*. 23*lb*. 13*oz*.

Ex. 4. In 54720878400 *Seconds*, How many *Days*, *Weeks*, and *Years*?

Anſwer, 1734 *Julian Years*.

SCHOLIUM IX.

SCHOLIUM IX.

340. When you are to divide by a Monome or by 12, as in the foregoing Example, where it is required to bring 802271 Farthings into Pence, Shillings, and Pounds *Sterling*: The Operation may be abbreviated, as in the Margin. (In. 319.320.321.322.)

$\frac{1}{4}$	802271 *qrs.*
$\frac{1}{12}$	200567 *d.* 3 *qr.*
$\frac{1}{20}$	1671 | 3 *s.* 11 *d.*
	835 *l.* 13 *s.* 11 *d.* 3 *qe.*

SCHOLIUM X.

341. The principal Use of *Reduction* is, in Questions of the Rule of Three, to bring Numbers of different Denominations into the same Denomination, and so fit them for Multiplication and Division (In. 199) As in the following Examples.

Example 1.

Suppose 94 Yards of Cloth cost in all 87 *l.* 13 *s.* 4 *d.* for 32 Ells of which were given after the Rate of 1 *l.* 6 *s.* 8 *d.* *per* Ell, it is required to determine what was given *per* Yard for the Remainder. This Question when all the Terms are brought to the same Denomination will be reduced to the following one. Suppose 376 Quarters of Cloth cost 21040 Pence, for 160 Quarters of which were given after the Rate of 320 Pence *per* 5 Quarters (or 64 Pence *per* Quarter) it is required to determine what was given *per* Yard (or *per* 4 Quarters) for the Remainder?

The Answer to which Question is performed by two Operations of the Rule of Three Direct, as follows.

First 1 *qr* : 64 *d.* = 160 *qr* : 10240 *d* the Price of the 32 Ells or 160 Quarters; then 376 *qrs* — 160 *qrs* = 216 *qrs* : and 21040 *d.* — 10240 *d.* = 10800 *d.* *per* Question: Therefore Secondly, 216 *qrs.* 10800 *d.* = 4 *qrs.* 200 *d.* or 16 *s.* 8 *d.* for the Price of 1 Yard of the Remainder: The Answer required.

Example 2.

If 16 *s.* 10 *d.* be paid for the Carriage of 2 *C.* 3 *qrs.* 14 *lb.* Averdupois, 75 Miles, What must be paid for the Carriage of 15 *C.* 1 *qr.* 22 *lb.* for 32 Miles, at the same Rate? Or

If 202 *d.* be paid for the Carriage of 322 *lb.* 75 Miles, What must be paid for the Carriage of 1730 *lb.* 32 Miles? A Question in the Compound Rule of Three Direct.

Answer, 322 *lb.* × 75 *m* : 1730 *lb.* × 32 *m.* = 202 *d.* : $463\frac{127}{2415}$ *d.* or 1 *l.* 18 *s.* $7\frac{127}{2415}$ *d.* the Sum to be paid (In. 198.)

Example

Example 3.

If when Wheat is sold at $6s.$ $8d.$ *per* Bushel; the Penny Loaf ought to weigh $9 oz.$ Troy, it is required to determine the Weight a of a Six Penny Loaf, when the Bushel of Wheat is sold for $10s.$ A Question in the Compound Rule of Three Inverse.

For $6s.$ $8d.$ put $80d.$ and for $10s.$ put $120d.$ (In. 292.) then by the single Rule of Three Direct (In. 196.) say

$1d. : 6d. = 9oz. : \frac{9 \times 6}{1} oz.$ the Weight of a Six Penny Loaf when Wheat is sold for $6s.$ $8d.$ or $80d.$ *per* Bushel. But the Weight $a oz.$ of a Six Penny Loaf (when Wheat is at $10s.$ or $120d.$ *per* Bushel) must be in Proportion so much less than the Weight $\frac{9 \times 6}{1} oz.$ of a Six Penny Loaf (when Wheat is at $80d.$ *per* Bushel) as $80d.$ is less than $120d.$ i. e. $120a$ must equal $80 \times \frac{9 \times 6}{1}$ $= 4320$; or $120d : 80d. = \frac{9 \times 6}{1} oz : 36 oz.$ (In. 190.) Therefore $a = 36 oz.$ the Answer required.

SCHOLIUM XI.

342. By Reduction also is determined the Value of the Coins, Weights, Measures, *&c.* of one Country in Comparison with another.

Example 1.

The Ell *Flemish* contains $\frac{3}{4}$ of our Yard, it is required then to determine how many Ells *English* are contained in 435 Ells *Flemish*.

Answer, $\frac{435 \times 3}{5} = 261 =$ the English Ells in 435 Ells Flemish.

Example 2.

The *Florence Crown* courrant is equal to $5s.$ $03d.$ English; I demand how many *Florence Crowns* are contained in $635l.$ $15s.$ $6d.$ For $5s.$ $3d.$ put $63d.$ and for $635l.$ $15s.$ $6d.$ put $152580d.$

Then $\frac{152580}{63} = 2422$ *Florence Crowns*.

Example 3.

How many *Portugal Testoons* at $1s.$ $3d.$ *per* Piece are contained in 75 *French Crowns* at $4s.$ $6d.$ *per* Crown?

$1s.$ $3d = 15d.$ and $4s.$ $6d = 54d.$ then $75 \times 54 = 4050d.$ the Number of *English Pence* in 75 *French Crowns*, which divided by $15d.$ gives 270 the Number of *Testoons* required.

Example 4.

The Proportion of the *London Foot* to the *Paris Foot* is nearly, as 15 to 16; many *London Feet* then are contained in 240 *Paris Feet*?

$15 : 16 = 240 : 256$ the *Answer* required.

I. A TABLE of Foreign Coins compar'd with English.

	Foreign Coins.		*English Coins.* l.	s.	d.	qr.
French	DENIER	is equal to	00	00	00	0 3/10
	Double Denier		00	00	00	0 3/5
	Liard		00	00	00	0 9/10
	Sols *Paris*		00	00	00	3 3/5
	Livre, Acc.		00	01	06	0
	Ecus Crown		00	04	06	0
	Pistole		00	16	00	0
Spanish	Maravedis old		00	00	00	0 3/4
	Quarta		00	00	00	2 [illegible]
	Octavo		00	00	01	1 [illegible]
	Real, *old Plata*		00	00	06	2 [illegible]
	Pieces of Eight, or *Piastra*		00	04	04	2
	Pistole		00	17	06	0
Portuguese	Kez		00	00	00	0 [illegible]
	Vintain		00	00	03	0 [illegible]
	Cruzada		00	05	04	3 [illegible]
	Mi-moeda, or half Pistole		00	13	06	0
	Mi-moeda d'Oro, or Pistole		01	07	00	0
	Doppio Moeda, or Double Pistole		02	14	00	0
	Ducat of fine Gold		06	15	00	0
Dutch	Penny		00	00	00	0 [illegible]
	Duyt		00	00	00	0 [illegible]
	Gros		00	00	00	2 [illegible]
	Stuyver, or Shilling common		00	00	01	0 [illegible]
	Scalin, or Shilling groſs		00	00	07	0
	Florin, or Guilder		00	02	00	0
	Dollar, or Ducatoon		00	06	00	0
Flemish	Patard, or Penny		00	00	00	0 [illegible]
	Groat		00	00	00	2 1/2
	Single Stiver		00	00	01	1
	Shilling		00	00	07	2
	Gulden		00	02	00	0
	Rixdollar		00	04	06	0
	Imperial		00	11	03	0

Germain

Foreign Coins.		English.			
		l.	s.	d.	qr.
Germain	Fenin is equal to	00	00	00	0 $\frac{275}{384}$
	Creux, or Kreuxer	00	00	00	2 $\frac{59}{108}$
	Dollar	00	01	03	1 $\frac{9}{5}$
	Obolus	00	02	03	2
	Izelotte	00	02	09	0
	Gulden, or Florin	00	03	00	0
	Rixdollar, or Dollar	00	04	06	0
	Ducat	00	07	06	0
Italian	Pignatelle = $\frac{1}{2}$ a Bayoco = $\frac{1}{5}$ a Julio				
	Julio	00	00	06	0 $\frac{7}{25}$
	Carlin	00	00	06	0
	Derlingue = $\frac{1}{4}$ a Justine	00	01	02	1
	Justine	00	04	09	0
	Monaco = a Croisat	00	04	04	0
	Sequin	00	09	02	0
Polish	Roup	00	00	04	3
	Abra	00	01	00	0 $\frac{4}{5}$
	Groch	00	00	00	2 $\frac{1}{2}$
Danish	Horse	00	01	01	2
	Marc Lubs	00	01	06	0
	Schesdale	00	03	00	0
Swedish	Christine	00	01	01	2
	Caroline	00	01	05	1
	Mark	00	00	02	1 $\frac{1}{3}$
	Roustic	00	00	02	1 $\frac{1}{8}$
	Alleuvre	00	00	00	0 $\frac{7}{12}$
Muscovite	Muskoske	00	00	00	1
	Polusk	00	00	00	2
	Copec of Silver, or Danaing	00	00	01	0
	Copec of Gold	00	01	06	0 $\frac{4}{11}$
Turkish	Aspre	00	00	00	2 $\frac{2}{3}$
	Shakee of *Aleppo* and *Scanderoon*	00	00	03	2
	Para Parat, or Parasi	00	00	01	2
	Sultanin Scherif, or Sequin	00	09	00	0

II. *A*

II. *A* TABLE *of* Foreign Weights *compar'd with those of* Amsterdam.

An Hundred Pound Weight of Amsterdam *is equal to*

lb		lb	
108	of *Alicant.*	163	of *Genoa*, Cash Weight.
105	of *Antwerp.*	102	of *Hamburg.*
120	of *Archangel*, or three *Poedes.*	106	of *Leyden.*
105	of *Arschot.*	105	of *Leipsic.*
120	of *Avignon.*	$105\frac{1}{2}$	of *Liege.*
98	of *Basil* in *Switzerland.*	114	of *Lisle.*
100	of *Bayonne* in *France.*	116	of *Lyons*, City Weight
166	of *Bergammo.*	$106\frac{1}{2}$	of *Lisbon.*
97	of *Berg ap Som.*	143	of *Legborn.*
$95\frac{1}{4}$	of *Bergen* in *Norway.*	109	of *London*, Averdupois.
111	of *Bern.*	105	of *Lovaine.*
100	of *Besanzon.*	105	of *Lubec.*
100	of *Bilboa.*	$141\frac{1}{2}$	of *Lucca*, Light Weight.
105	of *Bois le duc.*	114	of *Madrid.*
151	of *Bologne.*	105	of *Malines.*
100	of *Bourdeaux.*	$123\frac{1}{2}$	of *Marseilles.*
104	of *Bourg en Bresse.*	154	of *Messina*, Light Weight.
103	of *Bremen.*	168	of *Milan.*
125	of *Breslaw.*	120	of *Montpellier.*
105	of *Bruges.*	125	Bercheroćts of *Muscovy.*
105	of *Brussels.*	100	of *Nantes.*
105	of *Cadiz.*	106	of *Nancy.*
105	of *Cologne.*	169	of *Naples.*
125	of *Coningsberg.*	98	of *Nuremberg.*
$107\frac{1}{2}$	of *Copenhagen.*	100	of *Paris.*
87	Rottes of *Constantinople.*	$112\frac{1}{2}$	of *Revel.*
$113\frac{1}{2}$	of *Dantzic.*	109	of *Riga.*
100	of *Dort.*	100	of *Rochel.*
97	of *Dublin.*	146	of *Rome.*
97	of *Edinburgh.*	100	of *Rotterdam.*
143	of *Florence.*	96	of *Rouen*, Vicounty Weight.
98	of *Frankfort*, on the *Maine.*	100	of *S. Malo.*
105	of *Ghent.*	100	of *S. Sebastian.*
89	of *Geneva.*	$158\frac{1}{2}$	of *Saragosa.*

106 of

106 of *Seville*.
114 of *Smirna*.
110 of *Stetin*.
81 of *Stockholm*.
118 of *Tholouse* and *Upper Languedoc*.
151 of *Turin*.
$158\frac{1}{4}$ of *Valencia*.
182 of *Venice*, Small Weight.

III. *The Proportions of the* Long Measures *of several Nations to the* English *Foot, by Mr.* Greaves.

If the English *Standard Foot be divided into* 1000 *equal Parts, then*

Of such Parts
1068 = the *Paris* Royal Foot in the *Châtelet*.
1033 = the *Rhinland* Foot in *Snellius*.
$1007\frac{32}{100}$ = the *Greek* Foot.
967 = the *Roman* Foot, on the Monument of *Cossutius*.
972 = the *Roman* Foot, on the Monument of *Statilius*.
986 = the *Roman* Foot of *Villalpandus*, taken from the *Congius* of *Vespasian*.
1162 = the *Venetian* Foot.
2283 = the Ell of *Antwerp*.
2268 = the Ell of *Amsterdam*.
2260 = the Ell of *Leyden*.
687 = the Canna of *Naples*.
2760 = the Varra or Vare of *Almeria*, and *Gibralter*.
1913 = the Braccio of *Florence*.
815 = the Palm of *Genoa*.
1242 = the Common Braccio of *Sienna*.
1974 = the Braccio of *Sienna*, for Linnen.
732 = the Palm of the Architects at *Rome*, whereof ten make the Canna of the same Architects.
$695\frac{1}{10}$ = the Palm of the Braccio of the Merchants and Weavers at *Rome*; from a Marble in the Capitol, with this Inscription, *CURANTE LV POETO*.
2200 = the large *Turkish* Pique at *Constantinople*.
$2131\frac{1}{4}$ = the small *Turkish* Pique at *Constantinople*.
3197 = the Arish of *Persia*.
1824 = the Derah or Cubit of the *Egyptians*.

Problem XX.

343. To reduce a given Vulgar Fraction into a Decimal one.

Effection.

Add one or more Cyphers to the Numerator, and divide by the Denominator (according to the Directions In. 283) and the Quotient will be the Answer. *Ex. gr.* $\frac{1}{2} = \frac{1.0}{2} = 0.5$, $\frac{2}{3} = \frac{2.0}{3} = 6.6666$ &c. *ad infinitum*, $\frac{1}{12} = \frac{1.00}{12} = 0.083333$, &c. $\frac{3}{4} = \frac{3.0}{4} = .75$

Corollary XXII.

344. Hence it will be easy to find the decimal Parts equivalent to any given Part or Parts of Coin, Weight, Measure, &c. by considering them as vulgar Fractions.

Example 1.

Let it be required to find the Decimals equal to $5l.\ 12s.\ 6d.\ 2qrs.$ *Sterling* ($1l.$ being the Integer.)

Here $12s + 6d + 2qrs$ is equal to $\frac{12}{20} + \frac{6}{240} + \frac{2}{960}$ of a Pound (In. 292) But $\frac{12}{20} = 0.6$, $\frac{6}{240} = 0.025$, $\frac{2}{960} = 0.0020833$, &c. (In. 337.) Therefore $5l + 12s + 6d + 2qr. = 5l + 0.6l + 0.025l + 0.0020833$ &c. $l = 5.6270833$ &c.

Example 2. What are the Decimal Parts equal to $6s.\ 3d.\ 3qr.$ ($1s.$ being the Integer.)

Here $3d + 3qr = \frac{3}{12}s + \frac{3}{48}s$ (In. 292.) But $\frac{3}{12}s = 0.25s$, $\frac{3}{48} = 0.06251$ (In. 337.) Therefore $6s + 3d + 3qr = 6.3125s.$

And thus 2 Foot + $5\frac{1}{2}$ Inches $= 2\frac{5}{12} + \frac{1}{24} = 2\frac{11}{24}$ (In. 111) = 2.4175 Feet. (One Foot being the Integer.)

So also $9oz + 15\,p.w + 10\,gr.$ Troy Weight, or $\frac{9}{12} + \frac{15}{240} + \frac{10}{5760}$ (In. 293) $= 0.75\,lb + 0.0625\,lb + 0.001736111$ &c. *lb.* (In. 337.) $= 0.814236111$ &c. *lb.* Troy. (1 *lb.* Troy being the Integer.)

Scholium XII.

345. But for the more expeditious turning any Part or Parts of Coin, Weight, Measure, &c. into Decimals are made the following Tables, and such like.

Decimal TABLES.

1*l.* *Sterling* being the Integer.	
1 *s.*	= 0.05
1 *d.*	= 0.00416667
1 *qr.*	= 0.00104166

1 *C.* *Averdupois* being the Integer.	
$\frac{1}{4}$ *C.*	= 0.25
1 *lb.*	= 0.00892857
1 *oz.*	= 0.00055803

1 *Oz.*

Decimal TABLES.

1 Oz. *Troy* being the Integer.	1 *Day* of 24 *Hours* being the Integer.
1 *p.w.* = 0.05	1 *Hour* = 0.0416667
1 *gr.* = 0.00208333	1′ = 0.0006944
	1″ = 0.0000115

1 *l.* *Averdupois* being the Integer.

1 *oz.* = 0.0625

1 *dr.* = 0.0090625

The Use of which Tables is as follows.

Ex. gr. Suppose the Decimal Parts were required which are equal to 2 *Quarters*, 13 *lb* .05 *oz.* *Averdupois* Weight (1 *C* being the Integer.)

Here in it's proper Table you have the Decimals of 1 *Qr.* 1 *lb.* and 1 *oz.* whence 2 *qrs.* = 2 × 0.25 = 0.5, 13 *lb.* = 13 × 0.00892857 = 0.01160714, 5 *oz.* = 5 × 0.00055803 = 0.00279015. Therefore 2 *Qrs.* + 13 *lb.* + .05 *oz.* = 0.51439729 *C.* the *Answer* required, and so for any other.

PROBLEM XXI.

346. To reduce a given Decimal Fraction into a Vulgar one of a given Denomination: or which is the same thing; To find the Value of a given Decimal in the known Parts of Coin, Weight, Measure, *&c.*

Effection.

Multiply the given Decimal Fraction with the given Denominator of the vulgar Fraction whose Numerator is required, and the Product will be the said Numerator.

Ex. gr. Let it be required to find the Value of the Fractional Parts in 5.6270834 *l.* in Shillings, Pence, and Farthings.

Here the Integer is 1 *l.* *Sterling*, therefore the 5 in the Place of Integers is 5 *l.* then because 20 is the Number of Shillings in a Pound, therefore 0.6270834 *l.* is the same with $\frac{20 \times 0.6270834}{20}$ *l.* or $\frac{12.541668}{20}$ *l.* *i. e.* with 12.541668 *s.*

Again because 12 is the Number of Pence in a Shilling, therefore 0.541668 *s* is the same with $\frac{12 \times 0.541668}{12}$ *s* or $\frac{6.500016}{12}$ *s.* (In. 108) *i. e.* with 6.500016 *d.*

Lastly, because 4 is the Number of Farthings in a Penny, therefore 0.500016 *d.* is the same with $\frac{4 \times 0.500016}{4}$ *d.* or $\frac{2.000064}{4}$ *d.* *i. e.* with 2.000064 *qrs.* Consequently the value of the Fraction required is 5 *l.* 12 *s.* 6 *d.* 2 *qr.* near. See the Work in the Margin.

$$\begin{array}{r} 5|6270834 \\ 20 \\ \hline 12|541668|0 \\ 12 \\ \hline 6|500016 \\ 4 \\ \hline 2|000064 \end{array}$$

Ex. 2.

Ex. 2. What is the Value of 0,8142361 *lb.* Troy ?
The Operation is as follows.

```
    0|8142361 lb.
         × 12  = the Ounces in 1 lb. Troy.
   9|7708332  Oz.
        × 20  = the P.W. in 1 Oz.
  15|4166664  P.W.
        × 24  = the Gr. in 1 P.W.
    1666656
    833328
    9|999936  Gr. = 10 Grains nearly.
```

Therefore the Anſwer is 9 *Oz.* 15 *P.W.* 10 *Gr.* And by theſe two Examples it may be preſumed that the Learner will ſee how to anſwer any of the like kind.

Chap. VII.

Of the Rules of Practice.

Definition XXXI.

347. By *Practice* is generally meant thoſe compendious Rules uſed among Merchants and Tradeſmen in reſolving Queſtions of the ſingle Rule of Three Direct, when the firſt Term in the Proportion is Unity.

Problem XXII.

348. To reſolve Queſtions by the Rules of *Practice.*

This is only to be learned by Fxamples. The only general Precept, that can be given, is this, that the Learner be very expert in what is deliver'd concerning the Aliquot Parts of a Pound, Shilling, &c. (In. 292) and with the particular Tables of the Commodities he is to deal in.

Example 1. What is the Value of 7856 *lb.* Weight of any thing at a Farthing *per lb ?* i.e. If 1 *lb.* coſt 1 *qr.* What will 7856 *lb.* coſt at that Rate ?

Here becauſe 1 *qr.* is the 48th Part of a Shilling, therefore the 48 Part of 7856 *qrs.* or, which is the ſame, the $\frac{1}{4}$ of $\frac{1}{12}$ of 7856 *qrs.* will be the Anſwer required in Shillings, as follows.

$\frac{1}{4}$|7856 *qrs.*

$\frac{1}{6}$ | 7856 *qrs* = the Price of 7856*lb.* in Farthings.
$\frac{1}{8}$ | 1309. 2 *qrs* = the Price in Three Half-Pences and Farthings.
$\frac{1}{2|10}$ | 16|3 *s.* 8 *d* = the Price in Shillings and Pence.
8 *l.* 3 *s.* 8 *d* = the Price in Pounds, Shillings, and Pence.

Example 2. What comes 5963 *lb.* to, at 3 *qrs.* per *lb.*
Here 3 *qrs.* is the $\frac{1}{16}$ Part, or the $\frac{1}{4}$ of $\frac{1}{4}$ of a Shilling.

$\frac{1}{4}$ | 5963 the Price in three Farthings.
$\frac{1}{4}$ | 1490 .9 *qrs* = the Price in Three Pences and Farthings.
$\frac{1}{2|10}$ | 37|2 *s.* 8 *d.* 1 *qr* = the Price in Shillings, Pence, and Farthings.
18 *l.* 12 *s.* 8 *d.* 1 *qr* = the Price in Pounds, Shillings, &c.

Example 3. What comes 61543 *lb.* to, at 4 *d.* *per lb.*?
Here 4 *d.* is the $\frac{1}{3}$ Part of a Shilling. Therefore

$\frac{1}{3}$ | 61543 Fourpences = the Price in Fourpences.
$\frac{1}{2|10}$ | 2051|4 *s.* 4 *d* = the Price in Shillings and Pence.
1025 *lb.* 14 *s.* 4 *d* = the Price in Pounds, Shillings, and Pence.

Example 4. What comes 4793 *lb.* to, at 9 *d.* 3 *qrs.* per *lb.*?
Here the Price may be found first at 6 *d.* then at 3 *d.* and lastly at 3 *qrs.* and the Sum of all will make the Price at $9\frac{3}{4}$ *d.* as follows.

$\frac{1}{2}$ | 4793 *lb.* at $9\frac{3}{4}$ *d.* per *lb.*
$\frac{1}{2}$ | 2396 *s.* 6 *d* = the Price at 6 *d.* per *lb.*
$\frac{1}{4}$ | 1198 *s.* 3 *d* = the Price at 3 *d* per *lb.*
299 *s.* 6 *d.* 3 *qrs.* = the Price at 3 *qrs.* per *lb.*
} to be added together.

$\frac{1}{2|10}$ | 389|4 *s.* 3 *d.* 3 *qrs.* the Price at 9 $d\frac{3}{4}$ per *lb.* =.
194 . 14 *s.* 3 *d.* 3 *qrs.* the same in Pounds, Shillings, &c.

Example 5. What comes 7893 Yards to, at $10\frac{1}{4}$ *d.* per *Yard*?

$\frac{1}{3}$ | 7893 Yards at $10\frac{1}{4}$ *d per* Yard.
2631 *s.* = the Price at 4 *d.* *per* Yard.
2631 *s.* = the Price at 4 *d.* *per* Yard.
$\frac{1}{2}$ | 1315 *s.* 6 *d.* = the Price at 2 *d.* *per* Yard.
$\frac{1}{8}$ | 164 *s.* 5 *d.* 1 *d.* the Price at 1 *qr.* *per* Yard.
} to be added together.

$\frac{1}{2|10}$ | 674|1 *s.* 11 *d* 1 *d.* the Price at $10\frac{1}{4}$ *d.* *per* Yard in Shillings, Pence, &c.
337 *l.* 1 *s.* 11 *d.* 1 *d.* the same in Pounds, Shillings, &c.

Example 6. What comes 6578 Yards to, at 2 *s.* *per* Yard?

This is performed by cutting off the Figure in Unit's Place and doubling it thus

657|8 Yards at 2*s.* *per* Yard,
657 *l.* 16*s.* the *Anſwer* required.

The Reaſon of this is plain, ſince the Price of 6578 Yards at 1*s.* *per* Yard comes to 328*l.* 18*s.* (In. 339) the Double of which makes 657*l.* 16*s.* And ſo for any other. Whence if the given Price of the Integer be an even Number of Shillings it is but multiplying the Number of Integers by half that even Number of Shillings, doubling the firſt Figure in the Product for Shillings, and the reſt of the Product will be Pounds for the Anſwer required: As in the three enſuing Examples.

Ex. 7. What comes 2378 Yards to, at 12*s.* *per* Yard?
$\times 6 = \frac{12}{2}$
Anſwer 1426*l.* 16*s.*

Ex. 8. What comes 543 Ells to at 8*s.* *per* Ell?
$\times 4 = \frac{8}{2}$
Anſwer 217*l.* 4*s.*

Ex. 9. What comes 784 *Feet* to, at 18*s.* *per* Foot?
$\times 9 = \frac{18}{2}$
Anſwer 705*l.* 12*s.*

Ex. 10. What comes 157 *C.* Weight to, at 17*s.* per *C.*
157 *C.* at 17*s.* per *C.*
125*l.* 12*s.* } the Price at { 16*s.* } per *C.*
7*l.* 17*s.* } the Price at { 1*s.* } per *C.*
133*l.* 09*s.* The Price at 17*s.* per *C.*

Ex. 11. What comes 307 *Oz.* to, at 5*s.* per *Oz.*
$\frac{1}{4}$ | 307 *oz.* at 5*s.* per *Oz.*
Anſwer 76*l.* 15*s.*

Ex 12. What comes 755 *C.* to at 13*s.* 4*d.* per *C.*?
$\frac{1}{3}$ | 755 *C.* at 13*s.* 4*d.*
251*l.* 13*s.* 4*d.* } the Price at { 6*s.* 8*d.* } per *C.* add
251*l.* 13*s.* 4*d.* } the Price at { 6*s.* 8*d.* } per *C.* add
503*l.* 6*s.* 8*d.* the Price at 13*s.* 4*d.* per Cent.

Ex. 13. What comes 935 *C.* to, at 5*s.* 4*d.* per *C.*?
$\frac{1}{6}$ | 935 *C.* at 5*s.* 4*d.* per *C.*
155*l.* 16*s.* 8*d.* } the Price at { 3*s.* 4*d.* } per *C.* add
93*l.* 10*s.* 0*d.* } the Price at { 2*s.* } per *C.* add
249*l.* 6*s.* 8*d.* the Price at 5*s.* 4*d.* per *C.*

Ex. 14.

Ex. 14. What comes 353 Yards to, at 3 *l.* 9 *s.* 3 *d.* 3 *qr. per* Yard?

353 Yards at 3 *l.* 9 *s.* 3 *d.* 3 *qrs*

$\left\{\begin{array}{l} 88s.\ 3d. \\ 22s.\ \frac{1}{4}d. \\ 3177s. = 353\times 9 \end{array}\right\}$ the Price at $\left\{\begin{array}{l} 0s.\ 3d. \\ 0s.\ \frac{3}{4}d. \\ 09s.\ 0d. \end{array}\right\}$ to be added

$\frac{1}{20}$ 3287 *s.* 3 *d.* 3 *qr.*

$\left.\begin{array}{l} 164l.\ 7s.\ 3d.\ 3qrs. \\ 1059l. = 353\times 3 \end{array}\right\}$ the Price at $\left\{\begin{array}{l} 9s.\ 3d.\ 3qr. \\ 3l.\ 0s.\ 0d.\ 0qr. \end{array}\right\}$ to be added.

1223 *l.* 7 *s.* 3 *d* 3 *qr.* the *Answer* required.

Ex. 15. What comes 562½ Yards to, at 7 *s.* 6 *d. per* Yard?

562½ Yards at 7 *s.* 6 *d. per* Yard.

$\begin{array}{l} \frac{1}{4} \\ \frac{1}{2} \end{array}$ $\left\{\begin{array}{l} 140l.\ 10s. \\ 70l.\ \ 5s. \end{array}\right\}$ the Price of 562 Yards at $\left\{\begin{array}{l} 5s. \\ 2s.\ 6d. \end{array}\right\}$ add.

3 *s.* 9 *d.* the Price of the ½ Yard.

210 *l.* 18 *s.* 9 *d.* the *Answer* required.

Ex. 16. What comes 13 *C.* 3 *qrs.* 21 *lb.* to, at 1 *lb.* 6 *s.* 8 *d.* per *C.*

13 *C.* 3 *qrs.* 21 *lb.* at 1 *lb.* 6 *s.* 8 *d.*

Add
$\left\{\begin{array}{l} \left.\begin{array}{l} 13l. \\ 4.\ 6s.\ 8d. \end{array}\right\} \text{the Price of } 13C. \text{ at } \left\{\begin{array}{l} 1l.\ 0 \\ 0l.\ 6s.\ 8d. \end{array}\right. \\ 13s.\ 4d = \dfrac{1l + 6s + 8d}{2} = \text{the Price of 2 Qrs.} \\ 6s.\ 8d = \text{the Price of 1 Qr.} \\ 3s.\ 4d = \text{the Price of } \frac{1}{2} \text{ Qr. or } 14lb. \\ 1s.\ 8d = \text{the Price of } \frac{1}{4} \text{ Qr. or } 7\ lb. \end{array}\right.$

18 *l.* 11 *s.* 8 *d.* the *Answer* required.

C H A P.

Chap. VIII.

Of Evolution, or the Extraction of Roots.

Problem XXIII.

349. To find the Monome which is neareſt equal to the Root of any given Reſolvend beneath the Tenth Power.

Effection.

1. Make a Table of all the Powers beneath the Tenth for all the Nine Digits as follows, which may ſerve once for all.

Root, or firſt *Power.*	1	2	3	4	5	6	7	8	9
Square or 2d *Power.*	1	4	9	16	25	36	49	64	81
Cube or 3d *Power.*	1	8	27	64	125	216	343	512	729
Fourth Power.	1	16	81	256	625	1296	2401	4096	6561
Fifth Power.	1	32	243	1024	3125	7776	16807	32768	59049
Sixth Power.	1	64	729	4096	15625	46656	117649	262144	531441
Seventh Power.	1	128	2187	16384	78125	279936	823543	2097152	4782969
Eighth Power.	1	256	6561	65536	390625	2679616	5764801	16777216	43046721
Ninth Power.	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489

2. Diſtribute the given Reſolvend into Periods (In. 287, 288.)
3. From the Table above ſeek the neareſt *Homologous Power* to the higheſt Period, greater or leſſer than juſt.
4. Among the Digits in the firſt Line of the Table, ſeek the Root of that Homologous Power, which is the Figure or Monome required.

Ex. gr. Let it be required to find the Monome which is neareſt equal to the ſquare Root of 43046721. or $\overline{43}\,\overline{04}\,\overline{67}\,\overline{21}$ (In. 287, 288.) Here the higheſt Period is 43000000 the neareſt Square to which is 49000000 greater than juſt, whoſe Root 7000 is the Monome required. Whence the Square Root of 43046721 is ſome Number between 7000 and 6000 (In. 160, 290.)

Scholium XIII.

350. If the Reſolvend be higher than the 10th Power, the Table may be enlarged at pleaſure.

Problem XXIV.

351. To extract the Root from any given Reſolvend.

Effection.

Effection.

1. Find the Monome neareſt equal to the required Root, and note whether it be greater or leſſer than juſt (ln. 349, 350)
2. Subtract the *Homologous Power* of that Root from the given Reſolvend, if it be leſſer than juſt; or the Reſolvend from it, if greater; and call the Remainder the *firſt Dividend*.
3. Divide the Homologous Power by the Square of its Root.
4. Multiply that Quotient into half the Difference of the Exponent of the Power ſubtracted from the Square of the ſaid Exponent.
5. With that Product divide the *firſt Dividend*, and call the Quotient the *ſecond Dividend*.
6. Divide double the Root of the Homologous Power by the Exponent leſs one, and note the Quotient.
7. With that Quotient divide the *ſecond Dividend*, in ſuch manner, that the Quotient Figure be always twice added to the Diviſor, if the Homologous Power was taken leſs than juſt, or ſubtracted from it, if greater; *viz.* once, before it be multiplied into the Diviſor, and once with the next Quotient Figure following.
8. Obſerve in this Diviſion that for every Place of Figures annexed to the Diviſor, a Period of two Figures muſt be drawn from the Dividend; and for every Period drawn, beſides the firſt, a Cypher muſt be placed in the Quotient.
9. The Quotient thus found, added to the Root of the Homologous Power, when taken leſs than juſt, or ſubtracted from it, when greater, will be the Root required, if it be the *Square*; but, if it be the Root of any higher Power, exact only to the 3d, 4th, 5th, &c. Places of Figures, according as the Homologous Power is nearer, or farther from being equal to the given Reſolvend. Therefore in extracting the Roots from all Powers above the Square; if the Root be a Surd, or be not exactly found at the firſt Operation, then
10. A ſecond Operation muſt be made with the Homologous Power raiſed from ſo much of the Root as is already found true, or nearly true, proceeding with it, and the given Reſolvend as before, according as that Power is found to be greater or leſſer than juſt: And in this ſecond Operation the Root will be found true to 9 Places of Figures at leaſt, with which if need be a third Operation may be made, which will carry the Root at leaſt to 27 Places of Figures; a fourth Operation to 81 Places, &c. every following Operation ſtill tripling the foregoing one.

The Theorem by which this Resolution is made (as it is investigated (In. 504) is

$$r = b \pm \frac{\dfrac{d}{pb^{m-2}}}{\dfrac{m}{p}b \pm e}$$

Here b is put for the Root of the assumed Homologous Power, d for the first Dividend, m for the Exponent of the Power, e for the remaining Part of the Root, $p = m \times \frac{m-1}{2}$, $r = \overline{b \pm e}^{m} =$ the given Resolvend; and the Signs $+$ or $-$ are used according as b is lesser or greater than just. In particular

$$r^{\frac{1}{2}} = b \pm \frac{\dfrac{d}{b^0}}{2b \pm e} = \frac{d}{2b \pm e} \qquad r^{\frac{1}{5}} = b \pm \frac{\dfrac{d}{10b^3}}{\frac{2}{4}b \pm e} \qquad r^{\frac{1}{8}} = b \pm \frac{\dfrac{d}{28b^6}}{\frac{2}{7}b \pm e}$$

$$r^{\frac{1}{3}} = b \pm \frac{\dfrac{d}{3b^1}}{\frac{2}{2}b \pm e} \qquad r^{\frac{1}{6}} = b \pm \frac{\dfrac{d}{15b^4}}{\frac{2}{5}b \pm e} \qquad r^{\frac{1}{9}} = b \pm \frac{\dfrac{d}{36b^7}}{\frac{2}{8}b \pm e}$$

$$r^{\frac{1}{4}} = b \pm \frac{\dfrac{d}{6b^2}}{\frac{2}{3}b \pm e} \qquad r^{\frac{1}{7}} = b \pm \frac{\dfrac{d}{21b^5}}{\frac{2}{6}b \pm e} \qquad \text{\&c.} \qquad \text{\&c.}$$

Whence it appears that in extracting the Square Root, the third, fourth, and fifth Precepts of the foregoing Solution, are entirely struck out, the first and second Dividends being the same.

Example 1.

Let it be required to extract the Square Root from the given Resolvend $43046721 = r$.

Here the Square next less to the highest Period 43000000 is $36000000 = bb$ whose Root is $6000 = b$ less than just, Therefore $d = r - bb = 7046721$ the Dividend to be divided by $2b = 12000$ according to the Directions Pre. 7, and 8, as follows.

12000

12000 = 2*b*		
+ 500 = *e*		
12500 ..	)7046721(	561 = *e*
+ 560 = *e*	625	= 5 .. × 125 ..
13060	7967 ..	
+ 61 = *e*	7836 ..	= 6 . × 1306.
13121	13121	
	13121	= 1 × 13121
	(0)	

Whence the Root fought is 6000 + 561 = 6561, as may be proved by multiplying 6561 into it felf.

Example 2.

Let it be required to extract the Square Root from 43046721 with the affumed Square more than juft.

	49000000 = *bb*	*b* = 7000 more than juft.
	43046721 = *r*	
140 .. = 2*b*		
− 4 .. = *e*		
1360 .	)5953279 = *bb* − *r* = *d*	(439 = *e*
− 43 . = *e*	544	= 4 .. × 136 ..
1317 .	5132 ..	
− 39 = *e*	3951 ..	= 3 . × 1317 .
13131	118179	
	118179	= 9 × 13131
	(0)	

Whence the Root fought is 7000 − 439 = 6561 as above.

Example 3.

Extract the Square Root from the Surd Resolvend $1728 = r$.

$80 = 2b$	1728 (41.5692 &c. $= b + e$ the Root required,
$+1 = e$	$16.. = bb$
81	$)128 = r - bb$
$+1$	81
825	4700
$+5$	4125
83.06	57500
$+6$	49836
83.129	766400
$+9$	748161
83.1382	18239 &c. *ad infinitum.*
$+2$ &c.	

Example 4.

Let it be required to extract the Cube Root from the Resolvend $426.328706216447 = r$.

Here the nearest Cube to the highest Period 426, by the Table is $343 = bbb$ whose Root is $7 = b$ less than just. Therefore the Theorem is

$$r^{\frac{1}{3}} = b + \frac{\frac{d}{3b}}{b+e} : d = r - bbb = 83.328706216447 : \frac{d}{3b} = 3.968033 \text{ \&c.}$$

the second Dividend to be divided by $7 + e$.

$7 = b$	
$+.5 = e$	
7.5	) 3.968033 &c. (.527 $= e$
$+52 = e$	375
8.02	2180
$27 = e$	1604
8.047	57633
&c.	56329
	&c.

Hence the required Root is found at the first Operation to be 7.527 too much in the last Figure, but exact in the rest.

Where-

Wherefore assume $b = 7.527$ for a second Operation, whose Cube $b^3 = \overline{426}.\overline{447}\,\overline{672}\,\overline{183}$ greater than just.

$$\textit{Theorem, } r^{\frac{1}{3}} = b - \frac{\frac{d}{3b}}{b-e}$$

$d = b^3 - r = 118965966553$, $\frac{d}{3b} = 0.00526841$ the second Dividend.

7.527 = b
−.0007 = e
7.5263) 0.00526841 (0.0007 = e
526841
(0)

The true Root required is $7.527 - 0.0007 = 7.5263$. And if the Divisions be continued in this last Operation the Root thus found will continue true to the 12th Place of Figures, which is just triple the Figures found at the first Operation.

Example 5.

Let it be required to extract the First *Surfolid Root*, from the Number $\overline{32}.\overline{97158}\,\overline{93276}\,\overline{08832} = r$.

Here the nearest *Homologous Power* (by the Table) to the highest Period 32 is $32 = bbbbb$, whose Root is $2 = b$ less than just.

$$\textit{Theorem } r^{\frac{1}{5}} = b + \frac{\frac{d}{10b^3}}{\frac{1}{2}b + e}$$

$d = r - b^5 = 0.971589327608832$, $\frac{d}{10b^3} = 0.012144$ &c. for the second Dividend to be divided by $\frac{1}{2}b + e$, or $1 + e$.

1. = $\frac{1}{2}b$.
+0.01 = e
1.01) 0.012144 &c. (0.012 = e
+ 12 = e 101
1.022 2044
2044
(0)

The true Root is $2+0.012=2.012$, which by this Method is found exact to Six Places of Figures, at at the first Operation: The Reason is, because the assumed Power is equal to the highest Period of the Resolvend.

Scholium XIV.

352. After the same Manner may the Roots be extracted from any other Powers assignable, according to the foregoing general Theorem, from whence the Learner will easily see how to deduce particular ones at Pleasure: as is already done to his Hand to the 9th Power. And in the Extraction of Roots from all Powers whose Exponents are *Composit Numbers* (as from the *Biquadrate*, the *Sixth*, *Eighth*, *Ninth*, *Tenth*, *Twelfth*, &c. *Powers*) the Directions (In. 150) are to be observed.

Scholium XV.

353. The first Inventor of the Method of Division used in the *Seventh Precept* of the Effection of the last Problem was the Ingenious Mr. *John Ward* of *Chester*; but in the Extraction of Roots from all Powers above the *Square*, he erroneously adds or subtracts to or from the Divisor each Figure of e but once, instead of doing it twice: And so renders the Work less perfect than it wou'd be.

Problem XXV.

354. To extract the Root from any Power, in such sort, that 5 Figures, at least, but generally more, will come out true at the first Operation, 25 at the second, 125 at the third, &c. every following Operation quintupling the foregoing one.

Effection.

All Things being represented as before, the Effection of this Problem is performed by this general Theorem.

$$r^{\frac{1}{m}} = b \pm \cfrac{\cfrac{pd}{\overline{p^2-qm}|b^{m-2}}}{\cfrac{pm}{p^2-qm}b \pm \cfrac{qd}{\overline{p^2-qm}|b^{m-1}} \pm e}$$

Where $p = m \times \frac{m-1}{2}$, $q = m \times \frac{m-1}{2} \times \frac{m-2}{3}$.

Whence are formed the following particular *Theorems*.

$r^{\frac{1}{3}} =$

$$r^{\frac{1}{2}} = b \pm \frac{\frac{1d}{1b^0}}{\frac{6}{3}b \pm \frac{0d}{3b^2} \pm e} = b \pm \frac{d}{2b \pm e}$$

$$r^{\frac{1}{3}} = b \pm \frac{\frac{2d}{4b^1}}{\frac{6}{4}b \pm \frac{1d}{6b^3} \pm e} = b \pm \frac{\frac{d}{2b}}{\frac{3}{2}b \pm \frac{d}{6b^3} \pm e}$$

$$r^{\frac{1}{4}} = b \pm \frac{\frac{3d}{10b^2}}{\frac{6}{5}b \pm \frac{2d}{10b^4} \pm e} = b \pm \frac{\frac{3d}{10b^2}}{\frac{6}{5}b \pm \frac{d}{5b^4} \pm e}$$

$$r^{\frac{1}{5}} = b \pm \frac{\frac{4d}{20b^3}}{\frac{6}{6}b \pm \frac{3d}{15b^5} \pm e} = b \pm \frac{\frac{d}{5b^3}}{b \pm \frac{d}{5b^5} \pm e}$$

$$r^{\frac{1}{6}} = b \pm \frac{\frac{5d}{35b^4}}{\frac{6}{7}b \pm \frac{4d}{21b^6} \pm e} = b \pm \frac{\frac{d}{7b^4}}{\frac{6}{7}b \pm \frac{4d}{21b^6} \pm e}$$

$$r^{\frac{1}{7}} = b \pm \frac{\frac{6d}{56b^5}}{\frac{6}{8}b \pm \frac{5d}{28b^7} \pm e} = b \pm \frac{\frac{3d}{28b^5}}{\frac{3}{4}b \pm \frac{5d}{28b^7} \pm e}$$

$$r^{\frac{1}{8}} = b \pm \frac{\frac{7d}{84b^6}}{\frac{6}{9}b \pm \frac{6d}{36b^8} \pm e} = b \pm \frac{\frac{d}{12b^6}}{\frac{2}{3}b \pm \frac{d}{6b^8} \pm e}$$

$r^{\frac{1}{9}} =$

$$r^{\frac{1}{10}} = b \pm \frac{\frac{8d}{120b^7}}{\frac{6}{10}b \pm \frac{7d}{45b^{10}} \pm e} = b \pm \frac{\frac{d}{15b^7}}{\frac{3}{5}b \pm \frac{7d}{45b^{10}} \pm e}$$

$$r^{\frac{1}{11}} = b \pm \frac{\frac{9d}{165b^8}}{\frac{6}{11}b \pm \frac{8}{55b^{10}} \pm e} = b \pm \frac{\frac{3d}{55b^8}}{\frac{6}{11}b \pm \frac{8d}{55b^{10}} \pm e}$$

&c. &c. &c.

Example 1.

Let it be required to extract the Cube Root from the Number $\overline{897}\,\overline{289}.\overline{216}\,\overline{458}\,\overline{094}\,\overline{594}\,\overline{919}\,\overline{646}\,\overline{777}\,\overline{707} = r$

Here b^3 the nearest Cube to the highest Period is 1000000, whose Root is $100 = b$ more than just.

Theorem, $r^{\frac{1}{3}} = b - \dfrac{\frac{d}{2b}}{\frac{3}{2}b - \frac{d}{6bb} - e}$

$b^3 = 1000000.0$ $\quad \frac{d}{2b} = 513.5539$ &c. $\quad \frac{3}{2}b = 150.0$

$r = 897289.2$ &c. $\quad \frac{d}{6bb} = 1.71181$ &c.

$b^3 - r = d = 102710.7$ &c. $\quad \frac{3}{2}b - \frac{d}{6bb} = 148.288$ &c.

148.288

148,288		100.0000 $= b$
$-3 = e$		
145.288	)513.553(	3.5481 $= e$
-3.5	435.864	96.4519 $= b - e = r^{\frac{1}{3}}$ the true Root to
141.78	77 689	the 6th Place of Figures at the first Opera-
$-.54 = e$	70 894	tion.
141.2	6 795	
	5 649	
141.	1 145	
	1 129	
14	14	
	14	
	(0)	

Therefore for a second Operation assume $b = 96.4519$ whose Cube $b^3 = 897289.037003810359$ less than just.

$$\textit{Theorem},\ r^{\frac{1}{3}} = b + \cfrac{\frac{d}{2b}}{\frac{3}{2}b + \frac{d}{6bb} + e}$$

$r - b^3 = d = 0.17945428423591964677770 7.$

$\frac{d}{b} = 0.0018605572750347027562723699$ &c.

$\frac{d}{2b} = 0.00093027863751735137813 61849$ &c.

$\frac{\frac{d}{2b}}{b}$ or $\frac{d}{2bb} = 0.0000096450006429873478$ &c.

$\frac{d}{6bb} = 0.0000032150002143291159$ &c.

$\frac{3b}{2} = 144.67785$

$\frac{3b}{2} + \frac{d}{6bb} = 144.6778532150002143291159$ &c.

144.6778532150002143291159
+0.000006
144.6778592150002143291159)0.0009302786375173513781361849(0.00000642999999999999999998 &c. = e

+64 = e	8680671552900012859746954
144677865515000214329115	622114822273580921614995
+42 = e	578711462460000857316463
14467786603500021432911	43403359813500064298532
+29 = e	28935573807000042865823
1446778660640002143291	14467786606500021432709
+99 = e	13021007945760019289619
144677866073900214329	1446778660740002143090
+99 = e	1302100794665101928951
14467786607489021432	144677866074900214129
+99 = e	130210079467401192896
1446778660749892143	14467786607499021233
+99 = e	13021007946749029289
144677866074999114	1446778660749991944
+99 = e	1302100794674992028
14467786607500010	144677866074999916
+9 = e	130210079467500092
1446778660750001	14467786607499824
	13021007946750019
144677866075000	1446778660749805
	1302100794675001
14467786607500	144677866074804
	130210079467500
1446778660750	14467786607304
	13021007946750
144677866075	1446778660354
	1302100794675
14467786607	144677865879
	130210079467
1446778660	14467786412
	13021007946
144677866	1446778466
	1302100794
14467786	144677672
	130210079
1446778	14467593
	13021007
144677	1446587
	1302100
14467	144487
	130210
1446	14277
	13021
144	1256
	1156
	100

$$b = 96.4519$$
$$e = 0.00000642999999999999999998 \text{ \&c.}$$
$$b + e = 96.45190642999999999999999998 \text{ \&c.} = r^{\frac{1}{3}}$$ the Root required

required only too little by 2 at the 30th Figure. For the exact Root is 96.45190643 as may be tried by involving it to the third Power.

Example 2.

Let it be required to extract the first Surfolid Root from the Number 32.97158 93276 08832 $= r$.

Here b^5 the nearest *Homologous Power* to the highest Period is 32.0, whose Root is $2 = b$ less than just.

$$\textit{Theorem, } r^{\frac{1}{5}} = b + \frac{\frac{d}{5b^3}}{b + \frac{d}{5b^4} + e}$$

$d = r - b_5 = 0.971589327608832$, $\quad 5b_3 = 40$

$\frac{d}{5b^3} = 0.02428973$, &c. $\quad \frac{d}{5b^4} = 0.01214486$ &c.

2.01214486 &c. $= b + \frac{d}{5b}$

$+0.01 = e$ $\qquad$ 2.00000 &c. $= b$.

2.0221448)	0.024289733	(0.011999998 &c. $= e$.
$+11 = e$	20221448	
2.033144	4068284	
+19	2033144	
2.03504	2035140	
+99	1831540	
2.0360	203600	
+9	183243	
2.036	20357	
	18325	
2.03	2032	
	1832	
2.0	200	
	183	
2	17	
	16	
	1	

2.011999998 &c. $= r^{\frac{1}{5}}$ the Root only 2 too little at the 10th Figure at one Operation. For the exact Root is 2.012 as may be proved by involving it to the 5th Power.

Example 3.

Let it be required to extract the Root of the 365th Power from the given Number $1.05 = r$.

Here

Here $b^{\frac{1}{365}}$ the nearest Homologous Power to the highest Period is 1, whose Root is $1 = b$, less than just.

$$\textit{Theorem}, \; r^{\frac{1}{365}} = \frac{\dfrac{pbb}{p^2-365q\times b^{363}}}{\dfrac{365p}{p^2-365q}b + \dfrac{qd}{p^2-365q\times b^{364}}} + e$$

$$p = 365\times\frac{364}{2} = 66430,\; q = 66430\times\frac{363}{3} = 8038030$$

$$p^2 - 365q = 1479063950,\; \frac{p}{p^2-365q} = 0.000044913\,54143 \text{ \&c. } d = r$$

$$- b^{365} = 0.05,\quad \frac{pd}{p^2-365q} = \frac{pd}{p^2-365q\times b^{363}} = 0.000002245\,67707 \text{ \&c.}$$

$$\frac{365p}{p^2-365q}b = 0.0163934426 \text{ \&c. } \frac{q}{p^2-365q} = 0.0054345385 \text{ \&c.}$$

$$\frac{qd}{p^2-365q\times b^{364}} = 0.0002717269 \text{ \&c.}$$

$$0.0166651695 \text{ \&c.} = \frac{365p}{p^2-365q}b + \frac{qd}{p^2-365\times q\,b^{364}}$$

$+0.0001 = e$ $\qquad$ 1.00 &c. $= b$

0.0167651695) 0.000002245677707(0.0001336804 &c. $= e$

$+\;13 = e$ $\qquad$ 167651695 $\qquad$ 1.0001336804 &c. $= b + e =$

0.016895169 $\qquad$ 56916012

$+\;33$ $\qquad$ 50685508

0.01692816 $\qquad$ 6230504

$+\;36$ $\qquad$ 5078458

0.0169317 $\qquad$ 1152054

$+\;6$ $\qquad$ 1015905

0.016932 $\qquad$ 136149

$\qquad$ 135459

0.01693 $\qquad$ 690

$\qquad$ 677

$\qquad$ 23

$b^{\frac{1}{365}}$ the Root true (at the first Operation) to the 10th Figure; but too little by 3 at the 11th. And a second Operation performed after the same manner, with the above found Root put for b, will bring it to 50 Places of Figures at least. For which it will suffice to have, at most, the 50 highest Figures of these Powers b^{363}, b^{364}, b^{365}. But the first Operation is abundantly sufficient for Practice.

Scholium XVI.

355. For the Investigation of the *Theorem* whereby this new Method of *Extraction* is performed, see (In. 504.)

CHAP. IX.

Of LOGARITHMS.

DEFINITION XXXII.

356. ARTIFICIAL *Algorism* is the working by *Artificial Numbers*, called *Logarithms*, which are substituted in the Place of *Natural Numbers*, by means whereof *Multiplication* is performed by *Addition*, *Division* by *Subtraction*, and *Extraction of Roots* by dividing the given Resolvend by 2, 3, 4, *&c.* according as it is a *Square*, *Cube*, *Biquadrate*, &c.

DEFINITION XXXIII.

357. It has been shewn (In. 220) that any Geometrical Series whose first Term is Unity may be represented, if increasing above Unity, by

$$1 = r^0,\ r^1,\ r^2,\ r^3,\ r^4,\ r^5,\ r^6,\ r^7,\ r^8,\ \&c.$$

If decreasing below Unity, by

$$1 = r^0,\ r^{-1},\ r^{-2},\ r^{-3},\ r^{-4},\ r^{-5},\ r^{-6},\ r^{-7},\ r^{-8},\ \&c.$$

And the *Exponents* of every such Series are a Rank of Terms in ∸ whose first Term is 0 and common *Common Difference* is Unity (In. 221.)

If then the Terms in ∸ be set over their respective Terms in ∺ as in the two following Serieses (where *Ex. gr.* $r = 2$)

Arithmetical Progression	0 .	1 .	2 .	3 .	4 .	5 .	6 .	7 .	8 .	9	*&c.*
Geometrical Progression	1 .	2 .	4 .	8 .	16 .	32 .	64 .	128 .	256 .	512	*&c.*

The Terms in the former Series are stiled the *Logarithms* of their respective Terms in the latter: And the Terms in the latter are called the *Natural Numbers* of those *Logarithms*. Ex. gr. 0 is the *Logarithm* of the Number 1, 1 is the *Logarithm* of the Number 2, 2 of 4, 3 of 8, 4 of 16, *&c.* Where it is apparent

First, That the Sum of the *Logarithms* of any two or more Natural Numbers assumed as *Factors* is equal to the *Logarithms* of the *Product* made by those *Factors*, and *vice versa*, that the Difference made by subtracting the *Logarithm* of any one given Number from the *Logarithm* of another, is the *Logarithm* of the Quotient which arises from the Division of the latter Number from the former. (In. 153.)

Secondly, That the *Logarithm* of a Square is Double the *Logarithm* of its Root, of a Cube Triple, of a Biquadrate Quadruple, *&c.* and *vice versa*, that the *Logarithm* of the Square Root is one half the *Logarithm* of the Square,

of the Cube Root one third, of the Biquadrate Root one Fourth, &c. (In. 151, 152.)

Ex. gr. In the two Serieses above: Let the two Factors proposed be 4 and 32, whose *Logarithms* are 2 and 5; then $2+5=7$ the *Logarithm* of $128=4\times32$. So $5-2=3$ the *Logarithm* of $8=32\div4$.

Again, Suppose it were required to extract the Biquadrate Root from the Number 256 whose *Logarithm* is 8: The 4th of 8 or $\frac{8}{4}=2$ the *Logarithm* of 4; therefore 4 is the Biquadrate Root required. For if the Number 8, whose *Logarithm* is 3, to be involved to the third Power, then $3\times3=9$ the *Logarithm* of 512; therefore 512 is the Cube of 8.

Hypothesis V.

358. If the Scale of Powers 1, 10, 100, 1000, 10000, &c. have for their *Logarithms* 0, 1, 2, 3, 4, &c.
and consequently the Fractions 0.1, 0.01, 0.001, 0.0001, &c.
have for their *Logarithms* −1. −2. −3. −4. &c.
(In. 154, 220) then the *Logarithms* for all the intermediate natural Numbers between 1 and 10 will be some fractional Numbers between 0 and 1; the *Logarithms* for all the natural Numbers between 10 and 100 will be between 1 and 2, &c. In like manner the *Logarithms* for all the *Fractions* between 1 and 0.1 will be some Number affected with the Sign −, *i. e.* with some *Defective Number* between 0 and −1; the *Logarithms* for all the *Fractions* between 0.1 and 0.01 will be between −1 and −2, &c.

Hypothesis VI.

359. But because thus the *Logarithm* of no intermediate Natural Number can be exactly found, therefore they must in general be carried to such a certain Number of Fractions, as shall be thought sufficient for Use. And such a Number of Places is here 9, therefore the exact *Logarithms* are to be adapted accordingly, as in the following Serieses.

Logarithms	0.000000000,	1.000000000,	2.000000000,	3.000000000,	4.000000000,	&c.
Natural Numbers	1	10	100	1000	10000	&c.

Definition XXXIV.

360. The Figures to the Left-hand of the Point in every *Logarithm* are called the *Characteristicks* of those *Logarithms*. Thus 0 is the *Characteristick* of all the *Logarithms* to the Numbers between 1 and 10: 1 is the *Characteristick* of all the *Logarithms* to the Numbers between 10 and 100: 2 between 100 and 1000: 3 between 1000 and 10000, &c.

In

In like manner the *Characteristick* of all the *Logarithms* to all the Fractional Numbers between 1 and 0.1 is —0, between 0.1 and 0.01 is —1, between 0.01 and 0.001 is —2, between 0.001 and 0.0001 is —3, *&c.*

Problem XXVI.

361. To make a Table of *Logarithms* according to the *Hipotheses* (In. 358, 359) for all the Intermediate Numbers between 1 and 10, 10 and 100, 100 and 1000, 1000 and 10000.

And first to find the *Logarithms* for the Intermediate Numbers between 1 and 10, which *Ex. gr.* let be done by first finding the *Logarithm* of the Number 9.

Effection.

1. Make $A=1$, $B=10$ whose *Logarithms* by (In. 358, 359) are 0.00000000 and 1.00000000 as in the Table beneath.
2. Find a Mean *Geometrical Proportion* $C = 3.1622777$ between A and B (In. 200): also a Mean *Arithmetical Proportional* = 0.50000000 between their respective *Logarithms* (In. 178).: which will be the *Logarithm* of C.
3. Because C is much lesser than 9, find a Mean *Geometrical Proportional* $D = 5.6234132$ between B and C: Also find a Mean *Arithmetical Proportional* =0.75000000 between their respective *Logarithms*, which will be the *Logarithm* of D.
4. Because D is yet much lesser than 9, find a *Mean Geometrical Proportional* $E = 7.4989421$ between B and D: Also find a *Mean Arithmetical Proportional* = 0.87500000 between their respective *Logarithms*, which will be the *Logarithm* of E.
5. Continue thus finding *Mean Geometrical Proportionals*, so long as they either exceed or fall short of 9 by the Value of 0.0000001, according to the Number of Fractional Parts which the *Logarithms* are designed to consist of: Which will not happen till the twenty and sixth Trial, as in the following Table.

Mean Proportionals.	*Logarithms.*	*Mean Proportionals.*	*Logarithms.*
A 1.0000000	0.00000000	B 10.0000000	1.00000000
C 3.1622777	0.50000000	E 7.4989421	0.87500000
B 10.0000000	1.00000000	D 5.6234132	0.75000000
B 10.0000000	1.00000000	B 10.0000000	1.00000000
D 5.6234132	0.75000000	F 8.6596432	0.93750000
C 3.1622777	0.50000000	E 7.4989421	0.87500000

Mean

Mean Proportionals.	*Logarithms.*	*Mean Proportionals.*	*Logarithms.*
B 10.0000000	1.00000000	*R* 9.0002412	0.95425415
G 9.3057204	0.96875000	*S* 8.9999250	0.95423889
F 8.6596432	0.93750000	*P* 8.9996088	0.95422363
G 9.3057204	0.96875000	*R* 9.0002412	0.95425415
H 8.9768713	0.95312500	*T* 9.0000831	0.95424652
F 8.6596432	0.93750000	*S* 8.9999250	0.95423889
G 9.3057204	0.96875000	*T* 9.0000831	0.95424652
I 9.1398170	0.96093750	*V* 9.0000041	0.95424271
H 8.9768713	0.95312500	*S* 8.9999250	0.95423889
I 9.1398170	0.96093750	*V* 9.0000041	0.95424271
K 9.0579777	0.95703125	*X* 8.9999650	0.95424080
H 8.9768713	0.95312500	*S* 8.9999250	0.95423889
K 9.0579777	0.95703125	*V* 9.0000041	0.95424271
L 9.0173333	0.95507812	*Y* 8.9999845	0.95424217
H 8.9768713	0.95312500	*X* 8.9999650	0.95424080
L 9.0173333	0.95507812	*V* 9.0000041	0.95424271
M 8.9970796	0.95410156	*Z* 8.9999943	0.95424223
H 8.9768713	0.95312500	*Y* 8.9999845	0.95424217
L 9.0173333	0.95507812	*V* 9.0000041	0.95424271
N 9.0072008	0.95458984	*a* 8.9999992	0.95424247
M 8.9970796	0.95410156	*Z* 8.9999943	0.95424223
N 9.0072008	0.95458984	*V* 9.0000041	0.95424271
O 9.0021388	0.95434570	*b* 9.0000016	0.95424259
M 8.9970796	0.95410156	*a* 8.9999992	0.95424247
O 9.0021388	0.95434570	*b* 9.0000016	0.95424259
P 8.9996088	0.95422363	*c* 9.0000004	0.95424253
M 8.9970796	0.95410156	*a* 8.9999992	0.95424247
O 9.0021388	0.95434570	*c* 9.0000004	0.95424253
Q 9.0008737	0.95428467	*d* 8.9999998	0.95424250
P 8.9996088	0.95422363	*a* 8.9999992	0.95424247
Q 9.0008737	0.95428467	*c* 9.0000004	0.95424253
R 9.0002412	0.95425415	*e* 9.0000000	0.95424251
P 8.9996088	0.95422363	*d* 8.9999998	0.95424250

Therefore the *Logarithm* of 9 is 0.95424251 near.

Aft

6. After the same manner, if *Mean Proportionals* be found between *A* and *C* with their respective *Logarithms*, will be found the *Logarithm* of the Number 2: And so for any other Number.
7. The *Logarithms* of all *Composit Numbers*, or such as are compounded of other Numbers, will be had from the *Logarithms* of those Numbers of which they are compounded, and *vice versâ*; *Ex. gr.* the *Logarithm* of 9 doubled will give the *Logarithm* of 81; trebled will give the *Logarithm* of 729, &c. and halfed will give the *Logarithm* of 3. Again the *Logarithm* of 3 added to the *Logarithm* of 9 or subtracted from the *Logarithm* of 81 will give the *Logarithm* of 27: And if the *Logarithm* of 3 be added to the *Logarithm* of 729 it will give the *Logarithm* of 2187, if subtracted from it the *Logarithm* of 243, &c. And so for the *Logarithms* of any other Numbers.
8. When the *Logarithm* of any Number is got, the *Logarithms* of all Numbers on the same side of (*i. e.* either above or below) Unity which are in a ten-fold Ratio of it, are also had by only changing the *Characteristick* according to the Place of the highest Figure (In. 360). *Ex. gr.* the *Logarithm* of 2187, found as above, is 3.33984878; therefore the *Logarithm* of 21870 is 4.33984878 = the *Logarithm* of 2187 + the *Logarithm* of 10. The *Logarithm* of 218700 is 5.33984878: of 2187000 is 6.33984878: &c. The *Logaribm* of 218.7 is 2,33984878 = the *Logarithm* of 2187 — the *Logarithm* of 10. The *Logarithm* of 21.87 is 1.33984878: of 2.187 is 0.33984878.
9. Then for *Logarithms* of all Numbers in the same ten-fold Ratio below Unity subtract the *Logarithm* of that Number whose highest Figure is in Unite's Place from the *Logarithm* of 10, and the Difference with the Sign — before it will be the *Logarithm* of that Number which is 10 times less. *Ex. gr.* the *Logarithm* of 2.187 is 0.33984878, therefore the *Logarithm* of 0.2187 is —0.66015122: the *Logarithm* of 0.02187 is —1.66015122: of 0.002187 is —2.66015122: of 0.0002187 is —3.66015122: &c.

And thus may a Table of *Logarithms* be formed to all the Integers between 1 and 10000, and consequently by Pre. 9 to all the Fractions between 1 and 0.0001, which is usually stiled the *Canon* of *Logarithms*: And may be encreased at Pleasure. Q. E. E.

Problem XXVII.

362. To find the *Logarithm* of a Number greater than any contained in the *Canon*, but less than 10000000.

Effection.

1. Cut off as many Places to the Right-hand of the given Number as may make it not exceed 10000.

S 2. Find

2. Find the *Logarithm* of the Number so curtailed in the Table.
3. Subtract that *Logarithm* from the *Logarithm* of the next greater Number, and note the Difference.
4. Say by the Rule of Three, as 10, 100, 1000, or 10000 (according as the Places cut off from the given Number were 1, 2, 3, or 4) is to that Difference; so are the Figures cut off, to the Difference between the *Logarithm* of the Number curtailed, and the *Logarithm* of the given Number at Length, nearly
5. Let the Number so found be added to the *Logarithm* of the curtailed Number, and the Sum will be the *Logarithm* required: Observing its due *Characteristick* (In. 360.) Q. E. E.

Ex. gr. Let it be required to find the *Logarithm* of the Number 92375. Here the given Number needs but have one Figure cut off to bring it below 10000, the Number therefore curtailed is 9237, whose *Logarithm* in the Table is 3.9655309. The *Logarithm* of the next greater Number, *viz.* 9238, is 3.9655780. The Difference of the two *Logarithms* is 0.0000471. Then

$$10 : 471 = 5 : 235$$

Therefore $4.9655309 + 0.0000235 = 4.9655544$ is the *Logarithm* required.

PROBLEM XXVIII.

363. To find the *Logarithm* of a proper *Vulgar Fraction.*

Effection.

Subtract the *Logarithm* of the Numerator from the *Logarithm* of the Denominator, and prefix the Sign — to the Remainder, and it is done. *Ex. gr.* the *Logarithm* of $\frac{3}{7}$ is $0.8450980 - 0.4771213 = -0.3679767$.

PROBLEM XXIX.

364. To find the Number answering to a given *Logarithm*, which is not exactly contained in the Tables.

This Problem admits of two Cases.

Case 1.

If the *Characteristick* be 3; *i. e.* if the Number to which the given *Logarithm* agrees be between 1000 and 10000.

Effection.

1. Subtract the *Logarithm* of the next lesser Number from the *Logarithm* of the next greater, and also from the given *Logarithm*.
2. Say by the Rule of Three: As the former Difference is to Unity, so is the latter Difference, to the Excess of the Number sought above the next lesser Number nearly.

3. Add

5. Add that Excess to the lesser Number, and the Sum will be the Number sought. Q. E. E.

Ex. gr. Let it be required to find the Number answering to the *Logarithm* 3.7589982.

The next greater *Logarithm*	3.7590632	whose Numbers are	5742
The next lesser	3.7589875		5741
The former Difference	0.0000757		
The given *Logarithm*	3.7589982		
The next lesser	3.7589875		
The lesser Difference	0.0000107		

Then 757 : 1 = 107 : 0.14 near.

Therefore 5741.14 is the Number required.

Case 2.

If the *Characteristick* be 0, 1, or 2. *i. e.* if the Number to which the *Logarithm* agrees be between 1 and 1000, then the *Characteristick* is to be changed into 3; and the Number answering to such *Logarithm* sought for as above. And that Number, with one Place cut off, if the given *Characteristick* was 2, two Places if 1, three Places if 0, will be the Number required.

Ex. gr. If the above given *Logarithm* had been 2.7589982, its corresponding Number would have been 574.114: if 1.7589982, it would have been 57.4114: if 0.7589982, it would have been 5.74114, *&c.*

Problem XXX.

365. To find the Number answering to a given *Logarithm*, which is greater than any in the *Canon.*

Effection.

1. Subtract the *Logarithm* of the Number 10, 100, 1000, or 10000, till a lesser *Logarithm* be left than the last in the Table.
2. Seek the Number answering to that lesser *Logarithm* (In. 364.)
3. Multiply that Number by 10, 100, 1000, or 10000, and the Product is the Number sought. Q. E. E.

Ex. gr. Let the Number be sought which agrees to the *Logarithm* 7.7589982. Subtract from the given *Logarithm* 7.7589982 the *Logarithm* of 10000, which is 4.0000000, and the Remainder is 3.7589982, which by the last is the *Logarithm* of 5741.14. Therefore 5741.14×10000 = 57411400 is the Number sought.

Problem XXXI.

366. To find the Number answering to a given defective *Logarithm.*

Effection.

1. Let the *Logarithm* of 10000 be added to the defective *Logarithm*, i. e. let the latter consider'd as Positive, be subtracted from the former (In. 382, 383.)

2. Seek

2. Seek the Number agreeing to the Remainder, and that will be the Numerator of a Fraction whose Denominator is 10000. Q. E. E.

Ex. gr. Let the Fraction be sought agreeing to the defective *Logarithm* —0.3679767. Subtract +0.3679767 from 4.0000000, and the Remainder will be 3.6320233 the *Logarithm* of 4285.71. Therefore the Fraction sought is $\frac{428571}{1000000} = 0.428571$.

Problem XXXII.

367. From three Numbers given to find a Fourth in Geometrical Proportion (In. 196.)

Effection.

1. Add the *Logarithm* of the second and third Terms into one Sum.
2. From that Sum subtract the *Logarithm* of the first Term, and the Remainder will be the *Logarithm* of the fourth Term required.

Ex. gr. Let the given Terms be 4, 68, and 3.

$$
\begin{array}{rcll}
\textit{Log.}\text{ of } 68 & = & 1.8325089 & \\
\textit{Log.}\text{ of } 3 & = & \underline{0.4771213} & \\
\text{Sum} & = & 2.3096302 & \\
\textit{Log.}\text{ of } 4 & = & \underline{0.6020600} & \text{Subtract.} \\
\text{Remains } \textit{Log.}\text{ of } 51 & = & 1.7075702. &
\end{array}
$$

Therefore the fourth Term sought is 51.

Scholium XVII.

368. The first Author of this most useful Invention of *Logarithms* was the Noble Lord *Neper* of *Scotland* at *Edinburgh*, A. C. 1614. But the Form of *Logarithms* which we have was afterwards iuvented by him, and communicated to the learned *Henry Briggs* Professor of *Geometry* at *Oxford*, by whom a Canon was published at *London*, A. C. 1624, for all Numbers from 1 to 20000, and for eleven other *Chiliads*, viz. from 90000 to 101000: For all which Numbers he calculated the *Logarithms* to 14 Places of Figures.

Scholium XVIII.

369. The foregoing Method of treating upon *Logarithms*, from *Prob.* 26, is chiefly borrowed from the late excellent Edition of *Wolfius*, entitled *Elementa Matheseôs Universæ*.

The End of the Second PART.

ARITH-

ARITHMETICAL INSTITUTIONS.

PART III.

Of SPECIES ALGORISM.

CHAP. I.

General DEFINITIONS.

DEFINITION I.

370. SPECIES *Algorism* is the Method of handling Quantities in Specie according to *Affection*.

DEFINITION II.

371. By *Affection* of Quantity is here meant its contrary Procedure from Nothing, which is denoted by the Signs $+$ *Plus* or *More*, and $-$ *Minus* or *Less*: *Ex gr.* If $+a$ signifies so much Forwards, then $-a$ will signify the same Quantity Backwards. If $+a$ signify so much Upwards, $-a$ will signify the same Downwards. If $+a$ denote so much Stock, $-a$ will be the same in Debt, &c.

Partition I.

372. Quantities in respect of their *Affection*, are stiled *Positive* and *Defective*.

Definition III.

373. *Positive* or *Affirmative* Quantities are known by the Sign $+$ prefixed to them: as $+a$, $+b$, $+az$, $+27$.

Definition IV.

374. *Defective* or *Negative* Quantities are known by the Sign $-$ prefixed to them: as $-a$, $-b$, $-az$, -27.

Definition V.

375. Such Quantities as are of the *same Affection*, (*i. e.* are both *Positive* or both *Defective*) are called *Co-affected*; and such as are of *contrary Affections*, (*i. e.* the one *Positive* and the other *Defective*) *Contra-affected*.

Definition VI.

376. The *Balance* of two *Contra-affected* Quantities is that which remains after the Subtraction of the lesser from the greater, and is of the same Affection with the greater. *Ex gr.* If a Man have in Stock 7 *l.* and be in Debt 3 *l.* then the *Balance* of his Estate is 4 *l.* Stock, or 4 *l.* more than nothing: If he have in Stock 3 *l.* and be in Debt 7 *l.* the *Balance* of his Estate is 4 *l.* Debt, or 4 *l.* less than nothing. And if he have in Stock 7 *l.* and be in Debt 7 *l.* the *Balance* is nothing.

Postulatum.

377. That every Quantity may be considered either as *Positive* or *Defective*.

Axiom I.

378. If a Whole be *Positive* or *Defective*, all its Parts are so too.

Corollary I.

379. Therefore no *Positive* Quantity can be a Part of a *Defective* one, nor any *Defective* Quantity a Part of a *Positive* one; *i. e.* *Contra-affected* Quantities cannot be a Part one of another.

Corollary II.

380. Therefore only *Co-affected* Quantities can be added into one Sum or Whole. *Ex gr.*

To

To	$+a$	$+7a$	$-a$	$-7a$	$+a$	$-2a$
Add	$+a$	$+3a$	$-a$	$-3a$	$+b$	$-3b$
Sum	$+a+a$	$+7a+3a$	$-a-a$	$-7a-3a$	$+a+b$	$-2a-3b$
Or	$2a$	$+10$	$a-2a$	$-10a$	$+a+b$	$-2a-3b$

Definition VII.

381. The Sum of two *Contra-affected* Quantities is the ſame with the *Balance* of thoſe Quantities (*Inſt.* 376.) *Ex gr.*

To	$+a$	$+7a$	$-7a$	$+a$	$-2a$
Add	$-a$	$-3a$	$+3a$	$-b$	$+3b$
Sum is	$+a-a$	$+7a-3a$	$-7a+3a$	$+a-b$	$-2a+3b$
Or	0	$+4a$	$-4a$	$+a-b$	$-2a+3b$

Corollary III.

382. Hence it is evident, that to ſubtract a *Poſitive* Quantity, is the ſame as to add a *Defective*. *Ex. gr.*

From	$+a$	$+7a$	$+3a$	$-a$	$-7a$	$+2a$	$-2a$
Take	$+a$	$+3a$	$+7a$	$+a$	$+3a$	$+3b$	$+3b$
Rem.	$+a-a$	$+7a-3a$	$+3a-7a$	$-a-a$	$-7a-3a$	$+2a-3b$	$-2a-3b$
Or	0	$+4a$	$-4a$	$-2a$	$-10a$	$+2a-3b$	$-2a-3b$

Corollary IV.

383. And to ſubtract a *Defective* Quantity, is the ſame as to add a *Poſitive* one. *Ex. gr.*

From	$+a$	$+7a$	$+3a$	$-a$	$-7a$	$-3a$	$+a$
Take	$-a$	$-3a$	$-7a$	$-a$	$-3a$	$-7a$	$+b$
Rem.	$+a+a$	$+7a+3a$	$+3a+7a$	$-a+a$	$-7a+3a$	$-3a+7a$	$+a+b$
Or	$+2a$	$+10a$	$+10a$	0	$-4a$	$+4a$	$a+b$

Corollary V.

384. Therefore *Species Subtraction* is the ſame with *Addition*, only changing the Signs of the *Subtrahend*.

Corollary VI.

385. Hence it appears that *Quantity* has been treated of hitherto, only as *Poſitive*.

Axiom.

AXIOM II.

386. If a *Positive* Quantity be multiplied or divided by a *Positive* Quantity, the Product or Quotient will be *Positive*.

AXIOM III.

387. Any Whole ab, whether it be *Positive* or *Defective*, bears the same Ratio or Respect to its *Co-affected* Part a, *i. e.* $+ab$ contains or is contain'd in $+a$, the same Quantity of Times that $-ab$ contains or is contain'd in $-a$: or, which is all one, $\frac{+ab}{+a} = \frac{-ab}{-a}$ and $\frac{+a}{+ab} = \frac{-a}{-ab}$.

COROLLARY VII.

388. Therefore the Quotient, which expresses how often one *Defective* Quantity $-a$ is contained in another *Defective* Quantity $-ab$, must be a *Positive* Quantity. For $\frac{-ab}{-a} = \frac{+ab}{+a}$ by the last, and $\frac{+ab}{+a} = +b$ (*In.* 386.) Therefore $\frac{-ab}{-a} = +b$ (*In.* 21.)

COROLLARY VIII.

389. And consequently the Product, which is made by multiplying one *Defective* Quantity $-ab$ into another *Defective* Quantity $-\frac{1}{a}$ must be a *Positive* Quantity. For $\frac{-ab}{-a} = +b$ by the last, and $\frac{-ab}{-a} = -ab \times -\frac{1}{a}$ (*In.* 113.) Therefore $-ab \times -\frac{1}{a} = +b$ (*In.* 21.) Thus also $-a \times -b = +ab = +a \times +b$.

COROLLARY IX.

390. Again, if $\frac{-ab}{-a} = +b$ (*In.* 388.) then $-ab = +b \times -a$ (*In.* 83.) *i. e.* a *Positive* Quantity multiplied into a *Defective* one, or a *Defective* Quantity into a *Positive* one (*In.* 85.) will make a *Defective* Quantity in the Product.

COROLLARY X.

391. Also a *Defective* Quantity divided by a *Positive* one, or a *Positive* Quantity by a *Defective* one, will make a *Defective* Quantity in the Quotient. For $-ab = -a \times +b$ by the last; therefore $\frac{-ab}{+b} = -a$, or $\frac{-ab}{+a}$ =

$=-b$ (In. 77.) and $+a\times+b=-a\times-b$ (In. 389.) therefore $\frac{+a}{-b}=\frac{-a}{+b}$ or $+a:-b=-a:+b$ (In. 190.) consequently $\frac{+ab}{-b}=\frac{-ab}{+b}=-a$.

Corollary XI.

392. Whence we have this general Rule for Species Multiplication and Division, that *Co-affected* Quantities give a *Positive* Product or Quotient, and *Contra-affected* Quantities a *Defective* one; *i. e.*

$$\left\{\begin{matrix}+\times+\\+\div+\end{matrix}\right\}\text{and}\left\{\begin{matrix}-\times-\\-\div-\end{matrix}\right\}\begin{matrix}\text{give }+\\\text{in the}\end{matrix}\left\{\begin{matrix}\text{Product}\\\text{Quotient}\end{matrix}\right\}\begin{matrix}\textit{Ex.}\\\textit{gr.}\end{matrix}\left\{\begin{matrix}+ab=+a\times+b=-a\times-b\\+a=\frac{+ab}{+b}=\frac{-ab}{-b}\end{matrix}\right.$$

and

$$\left\{\begin{matrix}+\times-\\+\div-\end{matrix}\right\}\text{and}\left\{\begin{matrix}-\times+\\-\div+\end{matrix}\right\}\begin{matrix}\text{give }-\\\text{in the}\end{matrix}\left\{\begin{matrix}\text{Product}\\\text{Quotient}\end{matrix}\right\}\begin{matrix}\textit{Ex.}\\\textit{gr.}\end{matrix}\left\{\begin{matrix}-ab=+a\times-b=-a\times+b\\-a=\frac{+ab}{-b}=\frac{-ab}{+b}\end{matrix}\right.$$

Corollary XII.

393. Hence every Power, whose Exponent is an even Number, *i. e.* every *Square*, *Biquadrate*, *Sixth Power*, &c. must be *Positive*, whether the Root be *Positive* or *Defective* (In. 386.)

Corollary XIII.

394. Therefore the Root of every *Defective Square*, *Biquadrate*, *Sixth Power*, &c. is impossible; because it supposes two *Co-affected Factors* to give a *Defective Product*: And the Root of every *Defective Power*, whose Exponent is an odd Number, (*i. e.* of every *Defective Cube*, *Fifth*, *Seventh*, *Ninth*, &c. Power) is *Defective*.

Definition VIII.

395. Quantities designed by one Species are stiled *Monomes*, and those designed by more connected with the Signs $+$ and $-$, *Polynomes*.

Scholium I.

396. Every Quantity is supposed to be affected with one or other of these Signs (In. 377.) and whenever a Quantity has no Sign prefixed to it, as generally the leading Member in every *Polynome* has not, then is that Member understood to be affected with the Sign $+$. Thus $a+b-c$ is $+a+b-c$, b is $+b$, $d=+d$, &c.

Definition IX.

397. *Substitution* is the putting of one Expression in the room of another, and generally the more simple Expression in the room of a less simple one.

Ex. gr. in the *Polynome* $a+\overline{bb+2xz}^{\frac{1}{3}}+\overline{z-x}^{2}+\frac{x}{z}$ by ſubſtituting $p=\overline{bb+2xz}^{\frac{1}{3}}+\overline{z-x}^{2}+\frac{x}{z}$, we have this more ſimple Expreſſion $a+p$, which is equivalent to the former, and therefore may be uſed in its ſtead.

COROLLARY XIV.

398. Whence every *Polynome* by ſubſtituting a Letter for all the Members except the firſt, may be taken as a *Binomial.*

DEFINITION X.

399. And the Uſe of this *Species Arithmetic* in the Inveſtigation of Theorems, and Effection of Problems, is called *Algebra.*

CHAP. II.

Of Multiplication and Diviſion.

PROBLEM I.

400. TO multiply one *Polynome* into another.

Effection.

The *Effection* of this Problem in *Species* is directly as in Numbers, obſerving ln. 392. for the Signs. As follows.

$$\begin{array}{r}
a^2-3ab+bd \\
a-d \\
\hline
-a^2d+3abd-bd^2 \\
a^3-3a^2b+abd \quad \\
\hline
a^3-a^2d-3a^2b+4abd-bd^2
\end{array}
\qquad\qquad
\begin{array}{r}
x^3-x^2+x-1 \\
x+1 \\
\hline
x^3-x^2+x-1 \\
x^4-x^3+x^2-x \quad \\
\hline
x^4 \quad * \quad * \quad * -1
\end{array}$$

$$\begin{array}{r}
3x+2z \\
3x-2z \\
\hline
-6xz-4z^2 \\
9x^2+6xz \quad \\
\hline
9x^2 \quad * \quad -4z^2
\end{array}
\qquad\qquad
\begin{array}{c}
3z\overline{a-b}^{\frac{1}{2}} \\
\times\ \dfrac{2x}{3z} \\
\hline
\dfrac{6xz\overline{a-b}^{\frac{1}{2}}}{3z}=2x\overline{a-b}^{\frac{1}{2}}
\end{array}$$

q—

$$\overline{q - \frac{n^2 + n}{2}} \times 2 \qquad 2q - \overline{n^2 + n} = 2q - n^2 - n \text{ (In. 384.)}$$

$$\overline{q - \frac{n^2 - n}{2p}} \times 2p \qquad 2pq - \overline{n^2 - n} = 2pq - n^2 + n.$$

$$\frac{3x + 2z}{a^2 - 3ab + bd} \times \frac{3x - 2z}{a - d} = \frac{9x^2 - 4z^2}{a^3 - a^2d - 3a^2b + 4abd - bd^2} \text{ (In. 110.)}$$

PROBLEM II.

401. To divide by a *Polynome.*

Effection.

1. Let that Term in the Divisor which is contained in most Terms, and the oftnest in each Term, of the Dividend be put in the first Place, that which is the next often in the second Place, and so on.

2. Let the Terms in the Dividend be ordered accordingly; *viz.* Let those Terms which contain the first Term in the Divisor stand in the first Place, those which contain the second, in the second Place, and so on; observing in the first Place to set those Terms, which contain the first dividing Term oftnest, before those which contain it less often.

3. Set down the Divisor and the Dividend according to the Directions (*In.* 321) observing (*In.* 392) for the Signs. Examples follow.

$$a - d)\, a^3 \begin{smallmatrix} -a^2d \\ -3a^2b \end{smallmatrix} + 4abd - d^2b\, (a^2 - 3ab + db.$$
$$\underline{a^3 - a^2d}$$
$$* \; \underline{-3a^2b + 4abd}$$
$$\underline{-3a^2b + 3abd}$$
$$* \quad abd - d^2b$$
$$\underline{abd - d^2b}$$
$$* \quad *$$

$$x^3 - x^2 + x - 1)\, x^4 - 1\, (\quad x + 1$$
$$\underline{x^4 - x^3 + x^2 - x}$$
$$* \quad x^3 - x^2 + x - 1$$
$$\underline{x^3 - x^2 + x - 1}$$
$$* \quad * \quad * \quad *$$

$$3x + 2z)\, 9x^2 - 4z^2\, (3x - 2z$$
$$\underline{9x^2 + 6xz}$$
$$* \quad -6xz - 4z^2$$
$$\underline{-6xz - 4z^2}$$
$$* \quad *$$

$$3z\,|\overline{a - b}|^{\frac{1}{2}})\, 2x\,|\overline{a - b}|^{\frac{1}{2}}\, (\frac{2x}{3z}$$
$$\underline{2x\,|\overline{a - b}|^{\frac{1}{2}}}$$
$$*$$

$$q - \frac{n^2 + n}{2})\, 2q - n^2 - n\, (2$$
$$\underline{2q - n^2 - n}$$
$$* \quad * \quad *$$

$$q - \frac{n^2 - n}{2p})\, 2pq - n^2 + n\, (2p$$
$$\underline{2pq - n^2 + n}$$
$$* \quad * \quad *$$

$$\frac{9x^2-4z^2}{a^3-a^2d-3a^2b+4abd-bd^2} \div \frac{3x-2z}{a-d} = \frac{3x+2z}{a^2-3ab+bd} \quad \text{(In. 114.)}$$

The four next following Examples I have inserted out of Sir *Isaac Newton*'s *Universal Arithmetick*, p. 25, 26, 27.

$$-b+a)\,cbb\begin{matrix}-3ac\\-aa\end{matrix}b+a^3+2a^2c\,(-cb+2ac+aa.$$

$$cbb-acb$$

$$*\quad \begin{matrix}-2ac\\-aa\end{matrix}b+a^3+2a^2c$$

$$-2acb+2a^2c$$

$$-a^2b+a^3$$

$$-a^2b+a^3$$

$$*\quad *$$

$$y^2\begin{matrix}-a^2\\-c^2\end{matrix})\,y^6\begin{matrix}+a^2\\-2c^2\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}y^2-a^6-2a^4c^2-a^2c^4\,(y^4\begin{matrix}+2a^2\\-c^2\end{matrix}y^2+a^4+a^2c^2$$

$$y^6\begin{matrix}-a^2\\-c^2\end{matrix}y^4$$

$$*\quad\begin{matrix}+2a^2\\-cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}y^2.$$

$$\begin{matrix}+2a^2\\-cc\end{matrix}y^4\begin{matrix}-2a^4\\-a^2c^2\\+c^4\end{matrix}y^2$$

$$*\quad\begin{matrix}+a^4\\+a^2c^2\end{matrix}y^2-a^6-2a^4c^2$$

$$+a^4y^2-a^6-a^4c^2$$

$$a^2c^2y^2-a^4c^2-a^2c^4$$

$$a^2c^2y^2-a^4c^2-a^2c^4$$

$$*\quad *\quad *$$

$$y^2-2ay+a^2)\,y^4-3\tfrac{1}{2}a^2y^2+3a^3y-\tfrac{1}{2}a^4\,(y^2+2ay-\tfrac{1}{2}a^2$$

$$y^4-2ay^3+a^2y^2$$

$$*\quad 2ay^3-4\tfrac{1}{2}a^2y^2+3a^3y$$

$$2ay^3-4a^2y^2+2a^3y$$

$$*\quad -\tfrac{1}{2}a^2y^2+a^3y-\tfrac{1}{2}a^4$$

$$-\tfrac{1}{2}a^2y^2+a^3y-\tfrac{1}{2}a^4$$

$$*\quad *\quad *$$

a^2+

$$a^2+ab\overline{|2}^{\frac{1}{2}}+b^2)\,a^4 \quad * \quad * \quad +b^4\,(a^2-ab\overline{|2}^{\frac{1}{2}}+b^2$$

$$\underline{a^4+a^3b\overline{|2}^{\frac{1}{2}}+a^2b^2}$$

$$* \;-a^3b\overline{|2}^{\frac{1}{2}}-a^2b^2 \quad *$$

$$\underline{-a^3b\overline{|2}^{\frac{1}{2}}-2a^2b^2-ab^3\overline{|2}^{\frac{1}{2}}}$$

$$* \quad a^2b^2+ab^3\overline{|2}^{\frac{1}{2}}+b^4$$

$$\underline{a^2b^2+ab^3\overline{|2}^{\frac{1}{2}}+b^4}$$

$$* \quad * \quad *$$

$$x-1)\,x^4 \; * \; (x^3+x^2+x+1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}\ \&c.\ \text{or}\ x^3+x^2+x^1+x^0+x^{-1}+x^{-2}$$

$$+x^{-3}+x^{-4}\ \&c.=\frac{x^4}{x-1}$$

$$\underline{x^4-x^3}$$

$$x^3 \quad *$$

$$\underline{x^3-x^2}$$

$$x^2 \quad *$$

$$\underline{x^2-x}$$

$$x \quad *$$

$$\underline{x-1}$$

$$1 \quad *$$

$$\underline{1-\frac{1}{x}}$$

$$\frac{1}{x} \quad *$$

$$\underline{\frac{1}{x}-\frac{1}{x^2}}$$

$$\frac{1}{x^2} \quad *$$

$$\underline{\frac{1}{x^2}-\frac{1}{x^3}}$$

$$\frac{1}{x^3}\ \&c.\ \textit{ad infinitum}.$$

Here the Quotient is an *Infinite Geometrical Series*, either Decreasing or Increasing, according as x is greater or lesser than *Unity*, whose first Term is x^3, and common Divisor $=x$.

$$a+b\,)\;d\;*\;\left(\frac{d}{a}-\frac{db}{a^2}+\frac{db^2}{a^3}-\frac{db^3}{a^3}\text{ \&c. or } a^{-1}db^0-a^{-2}db^1+a^{-3}db^2-a^{-4}db^3\text{ \&c.}=\frac{d}{a+b}\right.$$

$$d+\frac{db}{a}$$

$$-\frac{db}{a}\quad *$$

$$-\frac{db}{a}-\frac{db^2}{a^2}$$

$$+\frac{db^2}{a^2}\quad *$$

$$+\frac{db^2}{a^2}+\frac{db^3}{a^3}$$

$$-\frac{db^3}{a^3}\text{ \&c. } \textit{ad infinitum.}$$

Here the Quotient consists of two *Infinite Series*, the Former having all its Terms Positive, whose first Term is $\frac{d}{a}$ or $a^{-1}db^0$; and common Multiplier $\frac{bb}{aa}$ or $a^{-2}b^2$; the latter having all its Terms Defective, whose first Term is $\frac{db}{a^2}$ or $a^{-2}db^1$, and common Mutiplier the same with the former.

$$a-b\,)\;d\;*\;\left(\frac{d}{a}+\frac{db}{a^2}+\frac{db^2}{a^3}+\frac{db^3}{a^4}\text{ \&c. or } a^{-1}db^0+a^{-2}db^1+a^{-3}db^2\text{ \&c.}=\frac{d}{a-b}\right.$$

$$d-\frac{db}{a}$$

$$\frac{db}{a}\quad *$$

$$\frac{db}{a}-\frac{db^2}{a^2}$$

$$\frac{db^2}{a^2}$$

$$\frac{db^2}{a^2}-\frac{db^3}{a^3}$$

$$\frac{db^3}{a^3}\text{ \&c. } \textit{ad infinitum.}$$

Here the Quotient is an *Infinite Geometrical Series*, whose first Term is $\frac{d}{a}$ or $a^{-1}db^0$, and common Multiplier is $\frac{d}{a}$ or $a^{-1}b^1$

In like Manner $\frac{a^{\frac{1}{2}}}{a-e}=\frac{1}{a^{\frac{1}{2}}}+\frac{e}{a|a^{\frac{1}{2}}}+\frac{ee}{aa|a^{\frac{1}{2}}}+\frac{eee}{aaa|a^{\frac{1}{2}}}$ &c.

SCHOLIUM II.

402. The Quotients of the four last Examples to the foregoing Problem are stiled *Infinitinomials*; where it may be observed, that if d in the second Example be put for the Numerator, and a for the Denominator less 1 of any Vulgar

Vulgar Fraction, and $b=1$, we will have all Vulgar Fractions, except $\frac{1}{2}$, expressed by an *Infinite Series*, thus:

$$\frac{1}{3}=\frac{1}{2+1}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}\ \&c. =\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}\ \&c.$$

$$\frac{2}{3}=\frac{2}{2+1}=\frac{2}{2}-\frac{2}{4}+\frac{2}{8}-\frac{2}{16}+\frac{2}{32}-\frac{2}{64}\ \&c. =1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}\ \&c.$$

$$\frac{3}{4}=\frac{3}{3+1}=\frac{3}{3}-\frac{3}{9}+\frac{3}{27}-\frac{3}{81}+\frac{3}{243}-\frac{3}{729}\ \&c. =1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}\ \&c.$$

&c. &c. &c. &c.

In like manner, if $b=1$ $d=1$, and $a=$ any Number greater than Unity in the third Example, then $\frac{d}{a-b}=\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\frac{1}{a^4}+\frac{1}{a^5}$ &c. which is a constant Law for expressing any Vulgar Fraction by an Infinite Series, whose Numerator is Unity, and Denominator a less 1.

And, if in the same Example, b still equal Unity, and d be put for the Numerator of a Vulgar Fraction, and a for the Denominator increased by Unity, then have we another universal Law for expressing Vulgar Fractions, as follows,

$$\left.\begin{array}{l} \frac{1}{2}=\frac{1}{3-1}=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\ \&c. \\ \frac{2}{3}=\frac{2}{4-1}=\frac{2}{4}+\frac{2}{16}+\frac{2}{64}+\frac{2}{256}\ \&c. \\ \frac{3}{4}=\frac{3}{5-1}=\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}\ \&c. \\ \&c.\ \&c.\ \&c. \end{array}\right\}\text{Therefore}\quad \begin{array}{l} \frac{2}{2}=1=\frac{2}{3}+\frac{2}{9}+\frac{2}{27}\ \&c. \\ \frac{3}{3}=1=\frac{3}{4}+\frac{3}{16}+\frac{3}{64}\ \&c. \\ \frac{4}{4}=1=\frac{4}{5}+\frac{4}{25}+\frac{4}{125}\ \&c. \\ \&c.\ \&c.\ \&c. \end{array}$$

COROLLARY XVI.

403. Whence it appears, that if a be put universally for any Integer, then

$$\frac{a-1}{a}+\frac{a-1}{a^2}+\frac{a-1}{a^3}+\frac{a-1}{a^4}+\frac{a-1}{a^5}\ \&c.=1$$

$$\frac{2a-2}{a}+\frac{2a-2}{a^2}+\frac{2a-2}{a^3}+\frac{2a-2}{a^4}+\frac{2a-2}{a^5}\ \&c.=2$$

$$\frac{3a-3}{a}+\frac{3a-3}{a^2}+\frac{3a-3}{a^3}+\frac{3a-3}{a^4}+\frac{3a-3}{a^5}\ \&c.=3$$

&c. &c. &c. &c. &c. &c.

Which gives an universal Law for expressing all Integers by an Infinite Series an infinite Number of Ways.

Otherwise

Otherwise any Integer may be put into an infinite Series of the same Value, according to the last Example of the foregoing Problem, by making $a=2$, and $b=1$, and $d=$ the given Integer; thus,

$$\frac{d}{a-b}=\frac{d}{2-1}=\frac{d}{2}+\frac{d}{4}+\frac{d}{8}+\frac{d}{16}+\frac{d}{32}\text{\&c.}$$

CHAP. III.

Of the Composition and Resolution of Powers.

PROBLEM III.

404. FROM the Binomial Root $b \pm e$ (*i. e.* $b+e$ or $b-e$) to raise any Power assignable.

Effection.

1. Put m for the Exponent of the required Power, by which Means the m Power of $b \pm e$ will be expressed by $\overline{b \pm e}^{m}$

2. Set down the Geometrical Series b^m b^{m-1} b^{m-2}. b^{m-3} b^{m-4} b^{m-5} &c. till the last Term be $b^0=1$.

3. Multiply each Term of the foregoing Series into each Term of this following one, *viz.* into $e^0=1$ e^1 e^2 e^3 e^4 e^5 &c. and the Series that results will be b^m $b^{m-1}e$ $b^{m-2}e^2$ $b^{m-3}e^3$ $b^{m-4}e^4$ $b^{m-5}e^5$ &c. till the last Term be $b^0e^m=e^m$

4. Connect all the Terms with the Sign $+$, if $b+e$ was the Root: And with the Signs $-+-+-+$ *&c.* interchangeably, if $b-e$ was the Root; thus $b^m \pm b^{m-1}e+b^{m-2}e^2 \pm b^{m-3}e^3+b^{m-4}e^4 \pm b^{m-5}e^5$ &c.

5. For the Coefficient of the first Term put 1, of the second Term $1\times\frac{m-0}{1}=m$, of the third Term $1\times\frac{m-0}{1}\times\frac{m-1}{2}=p$, of the fourth Term $1\times\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}=q$, of the fifth Term $1\times\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}=r$, of the sixth Term $1\times\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}=s$, *&c.* then will the required Power of $be \pm$ be as follows; $\overline{b\pm e}^{m}= b^m \pm\frac{m}{1}\times b^{m-1}e+\frac{m}{1}\times\frac{m-1}{2}b^{m-2}e\pm\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$

b^{m-}

$b^{m-3}e^3 + \frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}b^{m-4}e^4 \pm \frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}b^{m-5}e^5 +$
$\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}b^{m-6}e^6$ &c. $= b^m \pm mb^{m-1}e + pb^{m-2}e^2 \pm$
$qb^{m-3}e^3 + rb^{m-4}e^4 \pm sb^{m-5}e^5 + tb^{m-6}e^6$ &c.

Q. E. F. i. e.

$\overline{b\pm e}^1 = b\pm e$

$\overline{b\pm e}^2 = b^2\pm 2be+e^2$

$\overline{b\pm e}^3 = b^3\pm 3b^2e+3be^2\pm e^3$

$\overline{b\pm e}^4 = b^4\pm 4b^3e+6b^2e^2\pm 4be^3+e^4$

$\overline{b\pm e}^5 = b^5\pm 5b^4e+10b^3e^2\pm 10b^2e^3+5be^4\pm e^5$

$\overline{b\pm e}^6 = b^6\pm 6b^5e+15b^4e^2\pm 20b^3e^3+15b^2e^4\pm 6be^5+e^6$

$\overline{b\pm e}^7 = b^7\pm 7b^6e+21b^5e^2\pm 35b^4e^3+35b^3e^4\pm 21b^2e^5+7be^6\pm e^7$

$\overline{b\pm e}^8 = b^8\pm 8b^7e+28b^6e^2\pm 56b^5e^3+70b^4e^4\pm 56b^3e^5+28b^2e^6\pm 8be^7+e^8$

$\overline{b\pm e}^9 = b^9\pm 9b^8e+36b^7e^2\pm 84b^6e^3+126b^5e^4\pm 126b^4e^5+84b^3e^6\pm 36b^2e^7+9be^8\pm e^9$

&c. &c. &c. &c. &c. &c. &c. &c. &c. &c.&c.&c.

Otherwise the Powers of $b\pm e$ may be raised by Multiplication, according to (*In.* 400.) as follows;

$$\begin{array}{rl} & b+e \\ & b+e \\ \hline & be+e^2 \\ b^2+be & \\ \hline b^2+2be+e^2 & = \overline{b+e}^2 \\ & b+e \\ \hline & b^2e+2be^2+e^3 \\ b^3+2b^2e+be^2 & \\ \hline b^3+3b^2e+3be^2+e^3 & = \overline{b+e}^3 \end{array} \qquad \begin{array}{rl} & b-e \\ & b-e \\ \hline & -be+e^2 \\ b^2-be & \\ \hline b^2-2be+e^2 & = \overline{b-e}^2 \\ & b-e \\ \hline & -b^2e+2be^2-e^3 \\ b^3-2b^2e+be^2 & \\ \hline b^3-3b^2e+3be^2-e^3 & = \overline{b-e}^3 \end{array}$$

&c. &c.

Hence are deduced the following Corollaries.

Corollary XVI.

405. The Square of the Sum of any two Quantities is equal to the Sum of the Squares of the said two Quantities, together with twice the Product which arises by multiplying one into the other. The Cube of the Sum of any two Quantities is equal to the Sum of their Cubes, together with thrice the Product which arises by multiplying the lesser into the Square of the greater, and thrice the Product which arises by multiplying the greater into the Square of the lesser, &c. as in the Diagrams above.

Corollary XVII.

406. The Square of the Difference between any two Quantities is equal to the Sum of the Squares of the said Quantities less twice the Product which is made by multiplying one into the other. The Cube of the Difference between any two Quantities is equal to the Difference of their Cubes made less by the Difference between thrice the Product of the lesser into the Square of the greater, and thrice the Product of the greater into the Square of the lesser. And so for higher Powers, as in the foregoing Diagrams.

Corollary XVIII.

407. Every Power raised from a *Binomial Root* consists of one Term more than its Exponent.

Corollary XIX.

408. In the Scale of Powers whose Root is $b \pm e$, the Coefficients of each (which by the celebrated Mr. *Oughtred* are stiled their *Unciæ*) proceed thus. *1st*, The *Uncia* of every first Term is *Unity*. *2dly*, The *Uncia* of every second Term is the Exponent of the Power m. *3dly*, The *Uncia* of every third Term is a *Triangular Number*, whose Root or Side is $m-1$. *4thly*, The *Uncia* of every fourth Term is a *first Pyramidal Triangular*, whose Root is $m-2$. *5thly*, The *Uncia* of every fifth Term is a *second Pyramidal Triangular*, whose Root is $m-3$. *6thly*, The *Uncia* of every sixth Term is a *third Pyramidal Triangular*, whose Root is $m-4$, &c. From whence is grounded the last Precept in the foregoing Effection.

Corollary XX.

409. The *Unciæ* of the first and last Terms are the same, also the *Unciæ* of the second and last but one, of the third and last but two, of the fourth and last but three, &c.

Corollary XXI.

410. The Sum of all the *Unciæ* of any Power raised from a *Binomial* is equal to the Homologous Power of 2 or $1+1$. Thus, the Sum of the

the *Unciæ* of the *Square* is $1+2+1=2^2$ or 4: the Sum of the *Unciæ* of the *Cube* is $1+3+3+1=2^3$ or 8: the Sum of the *Unciæ* of the *Biquadrate* is $1+4+6+4+1=2^4$ or 16, &c.

Corollary XXII.

411. Because $b^{m-1}=\frac{b^m}{b}$, $b^{m-2}=\frac{b^m}{b^2}$, $b^{m-3}=\frac{b^m}{b^3}$, &c. (In. 154.) therefore $\overline{b\pm e}^m$ may be otherwise expressed by $b^m \pm m\frac{b^m e}{b}+p\frac{b^m e^2}{b^2}\pm q\frac{b^m e^3}{b^3}+r\frac{b^m e^4}{b^4}\pm s\frac{b^m e^5}{b^5}+t\frac{b^m e^6}{b^6}$ &c. If then we put $P=b$ and $Q=\frac{e}{b}$, so that $Q^2=\frac{e^2}{b^2}$, $Q^3=\frac{e^3}{b^3}$, $Q^4=\frac{e^4}{b^4}$ $Q^5=\frac{e^5}{b^5}$, $Q^6=\frac{e^6}{b^6}$, and $b\pm e=P\pm PQ$ we shall have $\overline{b\pm e}^m=\overline{P\pm PQ}^m=P^m\pm\frac{m}{1}P^m Q+\frac{m}{1}\times\frac{m-1}{2}P^m Q^2\pm\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}P^m Q^3+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}P^m Q^4 \pm\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}P^m Q^5+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}P^m Q^6$ &c. And if again, we make $P^m=A$, $\frac{m}{1}P^m Q=B$, $\frac{m}{1}\times\frac{m-1}{2}P^m Q^2=C$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}P^m Q^3=D$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times P^m Q^4=E$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}P^m Q^5=F$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}P^m Q^6=G$, &c. we shall have $\overline{b\pm e}^m=\overline{P\pm PQ}^m=A\pm\frac{m}{1}AQ+\frac{m-1}{2}BQ\pm\frac{m-2}{3}CQ+\frac{m-3}{4}DQ\pm\frac{m-4}{5}EQ+\frac{m-5}{6}FQ\pm\frac{m-6}{7}GQ$ &c. Which is called Sir *Isaac Newton*'s Theorem; this being one of the many Inventions found out by that great Man, and is said to be writ upon his Tomb-stone in *Westminster* Abbey. For the better understanding of which Theorem, take the following Example in Numbers.

Let

Let it be required to raiſe the Binomial $4+3$ to the fifth Power. Here $m=5$, $b=P=4$, $e=PQ=3$, therefore $Q=\frac{3}{4}$,

$$b^m=P^m=4^5=1024=A$$
$$\frac{m}{1}AQ=5\times1024\times\tfrac{3}{4}=3840=B$$
$$\frac{m-1}{2}BQ=2\times3840\times\tfrac{3}{4}=5760=C$$
$$\frac{m-2}{3}CQ=1\times5760\times\tfrac{3}{4}=4320=D$$
$$\frac{m-3}{4}DQ=\tfrac{1}{2}\times4320\times\tfrac{3}{4}=1620=E$$
$$\frac{m-4}{5}EQ=\tfrac{1}{5}\times1620\times\tfrac{3}{4}=243=F$$
$$\frac{m-5}{6}FQ=0\times243\times\tfrac{3}{4}=0=G$$

Therefore $16807=A+B+C+D+E+F$ is the Fifth Pow- of $4+3=7$. And ſo for any other. After the ſame manner alſo may the Root of any Number $b\pm e=P\pm PQ$ be reſolved by putting $\frac{1}{m}$ for m; *i. e.* by making $\frac{1}{2}=m$ for the Square Root, $\frac{1}{3}=m$ for the Cube, $\frac{1}{4}=m$ for the Biquadrate, *&c.*

PROBLEM IV.

412. To raiſe any *Polynome* to any Power aſſigned. *Ex. gr.* To raiſe the *Quadrinomial* $b+x+z+y$ to the Fourth Power.

Subſtitute $x+z+y=e$, and inſtead of $b+x+z+y$ we will have $b+e$ (In. 399.) then (In. 404.)

$e^2=x^2+2x|\overline{z+y}+\overline{z+y}^2=x^2+2xz+2xy+z^2+2zy+y^2$, $e^3=x^3+3x^2|\overline{z+y}+3x|\overline{z+y}^2+\overline{z+y}^3=x^3+3x^2z+3x^2y+3xz^2+6xzy+3xy^2+z^3+3z^2y+3zy^2+y^3$.

$e^4=x^4+4x^3|\overline{z+y}+6x^2|\overline{z+y}^2+4x|\overline{z+y}^3+z+y^4=x^4+4x^3z+4x^3y+6x^2z^2+12x^2zy+6x^2y^2+4xz^3+12xz^2y+12xzy^2+4xy^3+z^4+4z^3y+6y^2z^2+4zy^3+y^4$: Therefore, if theſe ſeveral Values of e, e^2, e^3, and e^4 be ſubſtituted in the *Formula* $b^4+4b^3e+6b^2e^2+4be^3+e^4$, we ſhall have $\overline{b+x+z+y}^4=b^4+4b^3x+4b^3z+4b^3y+6b^2x^2+12b^2xz+12b^2xy+6b^2z^2+12b^2zy+6b^2y^2+4bx^3+12bx^2z+12bx^2y+12bxz^2+24bxzy+12bxy^2+4bz^3+12bz^2y+12bzy^2+4by^3+x^4+4x^3z+4x^3y+6x^2z^2+12x^2zy+6x^2y^2+4xz^3+12xz^2y+12xzy^2+4xy^3+z^4+4z^3y+6zy^2+4zy^3+y^4$ for the Power required.

PROBLEM

Problem V.

413. To raiſe an *Infinitinomial* $A+By+Cy^2+Dy^3+Ey^4+Fy^5+Gy^6+Hy^7$ &c. to any Power aſſigned m.

Put $A=b$, and $By+Cy^2+Dy^3+Ey^4+Fy^5+Gy^6+Hy^7$ &c. $=e$.

$$\begin{array}{lll}
 & 1 & By+Cy^2+Dy^3+Ey^4+Fy^5+Gy^6 \text{ &c.} = e \\
 & 2 & By+Cy^2+Dy^3+Ey^4+Fy^5+Gy^6 \text{ &c.} = e \\
\hline
\text{Step 1} \times By & 3 & B^2y^2+BCy^3+BDy^4+BEy^5+BFy^6+BGy^7 \text{ &c.} \\
\text{Step 1} \times Cy^2 & 4 & \qquad BCy^3+CCy^4+CDy^5+CEy^6+CFy^7 \text{ &c.} \\
\text{Step 1} \times Dy^3 & 5 & \qquad\qquad BDy^4+CDy^5+DDy^6+DEy^7 \text{ &c.} \\
\text{Step 1} \times Ey^4 & 6 & \qquad\qquad\qquad BEy^5+CEy^6+DEy^7 \text{ &c.} \\
\text{Step 1} \times Fy^5 & 7 & \qquad\qquad\qquad\qquad BFy^6+CFy^7 \text{ &c.} \\
\text{Step 1} \times Gy^6 & 8 & \qquad\qquad\qquad\qquad\qquad BGy^7 \text{ &c.}
\end{array}$$

$$9 \quad B^2y^2+2BCy^3+\left.\begin{matrix}2BD\\CC\end{matrix}\right\}y^4+\left.\begin{matrix}2BE\\2CD\end{matrix}\right\}y^5+\left.\begin{matrix}2BF\\2CE\\DD\end{matrix}\right\}y^6+\left.\begin{matrix}2BG\\2CF\\2DE\end{matrix}\right\}y^7 \text{ &c.} = ee$$

$$By+Cy^2+Dy^3+Ey^4+Fy^5 \text{ &c.} = e$$

$$\begin{array}{lll}
\text{Step 9} \times By & 10 & B^3y^3+2B^2Cy^4+\left.\begin{matrix}2B^2D\\BC^2\end{matrix}\right\}y^5+\left.\begin{matrix}2B^2E\\2BCD\end{matrix}\right\}y^6+\left.\begin{matrix}2B^2F\\2BCE\\BDD\end{matrix}\right\}y^7 \text{ &c.} \\
\text{Step 9} \times Cy^2 & 11 & \qquad B^2Cy^4+2BC^2y^5+\left.\begin{matrix}2BCD\\C^3\end{matrix}\right\}y^6+\left.\begin{matrix}2BCE\\2CCD\end{matrix}\right\}y^7 \text{ &c.} \\
\text{Step 9} \times Dy^3 & 12 & \qquad\qquad B^2Dy^5+2BCDy^6+\left.\begin{matrix}2BDD\\CCD\end{matrix}\right\}y^7 \text{ &c.} \\
\text{Step 9} \times Ey^4 & 13 & \qquad\qquad\qquad B^2Ey^5+2BCE\,y^7 \text{ &c.} \\
\text{Step 9} \times Fy^5 & 14 & \qquad\qquad\qquad\qquad B^2F\,y^7 \text{ &c.}
\end{array}$$

$$15 \quad B^3y^3+3B^2Cy^4+\left.\begin{matrix}3B^2D\\3BC^2\end{matrix}\right\}y^5+\left.\begin{matrix}3B^2E\\6BCD\\CCC\end{matrix}\right\}y^6+\left.\begin{matrix}3B^2F\\6BCE\\3BDD\\3CCD\end{matrix}\right\}y^7 \text{ &c.} = eee$$

$$By+Cy^2+Dy^3+Ey^4 \text{ &c.} = e$$

Step 15×	By	16	$B^4y^4+3B^3Cy^5+\left.\begin{matrix}3B^3D\\3B^2C^2\end{matrix}\right\}y^6+\left.\begin{matrix}3B^3E\\6B^2CD\\BCCC\end{matrix}\right\}y^7$ &c.	
	Cy^2	17	$B^3Cy^5+3B^2C^2\ y^6+\left.\begin{matrix}3B^2CD\\3BCCC\end{matrix}\right\}y^7$ &c.	
	Dy^3	18	$B^3D\ y^6+3B^2CD\ y^7$ &c.	
	Ey^4	19	$B^3E\ y^7$ &c.	
		20	$B^4y^4+4B^3Cy^5+\left.\begin{matrix}4B^3D\\6B^2C^2\end{matrix}\right\}y^6+\left.\begin{matrix}4B^3E\\12B^2CD\\4BC^3\end{matrix}\right\}y^7$ &c.	$=eeee$
			$By+Cy^2+Dy^3$ &c.	$=e$
Step 20×	By	21	$B^5y^5+4B^4Cy^6+\left.\begin{matrix}4B^4D\\6B^3C^2\end{matrix}\right\}y^7$ &c.	
	Cy^2	22	$B^4Cy^6+4B^3C^2\ y^7$ &c.	
	Dy^3	23	$B^4D\ y^7$ &c.	
		24	$B^5y^5+5B^4Cy^6+\left.\begin{matrix}5B^4D\\10B^3C^2\end{matrix}\right\}y^7$ &c.	$=eeeee$
			$By+Cy^2$ &c.	$=e$
Step 24×	By	25	$B^6y^6+5B^5Cy^7$ &c.	
	Cy^2	26	B^5Cy^7 &c.	
		27	$B^6y^6+6B^5Cy^7$ &c.	$=eeeeee$
			By &c.	$=e$
Step 27×	By	28	B^7y^7 &c.	$=eeeeeee$

Then if in the *Formula* $b^m+mb^{m-1}e+pb^{m-2}e^2+qb^{m-3}e^3+rb^{m-4}e^4+sb^{m-5}e^5+tb^{m-6}e^6+vb^{m-7}e^7$ &c. (ln. 404.) for b be writ A, and for e, e^2, e^3, e^4, e^5, e^6, e^7, &c. be substituted their Values found as above, we shall have this *Formula*; for the m Power of the given Infinitinomial $A+By+Cy^2+Dy^3+Ey^4+Fy^5+Gy^6$ &c.

b^m

$$\left.\begin{array}{lllllll}
b^m = A^m & & & & & & \\
mb^{m-1}e = +mA^{m-1}By & +mA^{m-1}Cy^2 & +mA^{m-1}Dy^3 & +mA^{m-1}Ey^4 & +mA^{m-1}Fy^5 & +mA^{m-1}Gy^6 & +mA^{m-1}Hy^7 \ \&c. \\
pb^{m-2}e^2 = & pA^{m-2}B^2y^2 & +2pA^{m-2}BCy^3 & +\begin{matrix}2pA^{m-2}BD\\ pA^{m-2}C^2\end{matrix}y^4 & +\begin{matrix}2pA^{m-2}BE\\ 2pA^{m-2}CD\end{matrix}y^5 & +\begin{matrix}2pA^{m-2}BF\\ 2pA^{m-2}CE\\ pA^{m-2}D^2\end{matrix}y^6 & +\begin{matrix}2pA^{m-2}BG\\ 2pA^{m-2}CF\\ 2pA^{m-2}DE\end{matrix}y^7 \ \&c. \\
qb^{m-3}e^3 = & & +qA^{m-3}B^3y^3 & +3qA^{m-3}B^2C\,y^4 & +\begin{matrix}3qA^{m-3}B^2D\\ 3qA^{m-3}BC^2\end{matrix}y^5 & +\begin{matrix}3qA^{m-3}B^2E\\ 6qA^{m-3}BCD\\ qA^{m-3}C^3\end{matrix}y^6 & +\begin{matrix}3qA^{m-3}B^2F\\ 6qA^{m-3}BCE\\ 3qA^{m-3}BDD\\ 3qA^{m-3}CCD\end{matrix}y^7 \ \&c. \\
rb^{m-4}e^4 = & & & +rA^{m-4}B^4\,y^4 & +4rA^{m-4}B^3C\,y^5 & +\begin{matrix}4rA^{m-4}B^3D\\ 6rA^{m-4}B^2C^2\end{matrix}y^6 & +\begin{matrix}4rA^{m-4}B^3E\\ 12rA^{m-4}B^2CD\\ 4rA^{m-4}BC^3\end{matrix}y^7 \ \&c. \\
sb^{m-5}e^5 = & & & & sA^{m-5}B^5\,y^5 & +5sA^{m-5}B^4C\,y^6 & +\begin{matrix}5sA^{m-5}B^4D\\ 10sA^{m-5}B^3C^2\end{matrix}y^7 \ \&c. \\
tb^{m-6}e^6 = & & & & & tA^{m-6}B^6\,y^6 & +6tA^{m-6}B^5Cy^7 \ \&c. \\
vb^{m-7}e^7 = & & & & & & +vA^{m-7}B^7y^7 \ \&c. \\
\&c. & & & & & & \&c.
\end{array}\right\} =$$

$$= \overline{Ay^0 + By^1 + Cy^2 + Dy^3 + Ey^4 + Fy^5 + Gy^6 + Hy^7 \ \&c.}^{\,m}$$

COROLLARY

Corollary XXIII.

414. The ſame Formula will alſo ſerve for raiſing the m Power from the Infinitinomial $Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6$ &c. only inſtead of y^0 there will be y^m, $y^1=y^{m+1}$, $y^2=y^{m+2}$, $y^3=y^{m+3}$. *Ex. gr.* ſuppoſe $m=2$, then the Square of $Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6+Gy^7$ &c. will be

$$A^2y^2+2AB\,y^3+\begin{matrix}2AC\\BB\end{matrix}y^4+\begin{matrix}2AD\\2BC\end{matrix}y^5+\begin{matrix}2AE\\2BD\\CC\end{matrix}y^6+\begin{matrix}2AF\\2BE\\2CD\end{matrix}y^7+\begin{matrix}2AG\\2BF\\2CE\\DD\end{matrix}y^8+\begin{matrix}2AH\\2BG\\2CF\\2DE\end{matrix}y^9\ \&c.$$

Again, ſuppoſe $m=3$, then $\overline{Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6+Gy^7\ \&c.}^{\,3}$ will be

$$A^3\ y^3\pm3A^2B\ y^4+\begin{matrix}3A^2C\\3AC^2\end{matrix}y^5+\begin{matrix}3A^2D\\6ABC\\BBB\end{matrix}y^6+\begin{matrix}3A^2E\\6ABD\\3ACC\\3B^2C\end{matrix}y^7+\begin{matrix}3A^2F\\6ABE\\6ACD\\3B^2D\\3BC^2\end{matrix}y^8+\begin{matrix}3A^2G\\6ABF\\6ACE\\3ADE\\3B^2E\\6BCD\\CCC\end{matrix}y^9+\begin{matrix}3A^2H\\6ABG\\6ACF\\6ADE\\3B^2F\\6BCE\\3BDD\\3CCD\end{matrix}y^6\ \&c.$$

The Biquadrate of $Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6+Gy^7$ &c. will be

$$A^4y^4+4A^3By^5+\begin{matrix}4A^3C\\ \\6A^2B^2\end{matrix}y^6+\begin{matrix}4A^3D\\12A^2BC\\4AB^3\end{matrix}y^7+\begin{matrix}4A^3E\\12A^2BD\\6A^2C^2\\12AB^2C\\BBBB\end{matrix}y^8\ \&c.$$

&c. &c.

Corollary XXIV.

415. Conſequently if $A=B=C=D=E=F=G$ &c. then the Square of $Ay+Ay^2+Ay^3+Ay^4+Ay^5$ &c. or of $A\times y+y^2+y^3+y^4+y^5+y^6+y^7$ &c. will be $A^2\times y^2+2y^3+3y^4+4y^5+5y^6+6y^7+7y^8$ &c. its Cube will be $A^3\times y^3+3y^4+6y^5+10y^6+15y^7+21y^8+28y^9$, &c. its Biquadrate $A^4\times y^4+4y^5+10y^6+20y^7+35y^8+56y^9+84y^{10}$ &c. its fifth Power $A^5\times y^5+5y^6+15y^7+35y^8+70y^9+126y^{10}+210y^{11}$, &c. &c. Whence it appears that the Coefficients or *Unciæ* of the Powers of y in the Square are a Rank of Laterals, in the Cube a Rank of Triangular Numbers, in the Biquadrate a Rank of Pyramidals Triangular, in the fifth Power a Rank of ſecond Pyramidals Triangular, &c. *Ex. gr.*

If

If $A=1$ and $y=\frac{1}{2}$ then the Square of the Infinite Series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ &c. will be $\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}$ &c. its Cube $=\frac{1}{8}+\frac{3}{16}+\frac{6}{32}+\frac{10}{64}$ &c. its Biquadrate $=\frac{1}{16}+\frac{4}{32}+\frac{10}{64}+\frac{20}{128}$ &c. its fifth Power $=\frac{1}{32}+\frac{5}{64}+\frac{15}{128}+\frac{35}{256}$ &c. &c.

PROBLEM VI.

416. To extract the m Root from any Resolvend $b^m \pm mb^{m-1}e+pb^{m-2}e^2 \pm qb^{m-3}e^3+rb^{m-4}e^4 \pm sb^{m-5}e^5+tb^{m-6}e^6$ &c. and 1st from $b^m+mb^{m-1}e+pb^{m-2}e^2+qb^{m-3}e^3+rb^{m-4}e^4+sb^{m-5}e^5+tb^{m-6}e^6$ &c. $=r$.

Effection.

1. Find the Root b of the first Term b^m, and set it down to the Right Hand of the Resolvend, behind the Mark (.

2. Subtract b^m from the Resolvend, and the Remainder will be $+mb^{m-1}e+pb^{m-2}e^2+qb^{m-3}e^3+rb^{m-4}e^4+sb^{m-5}e^5+tb^{m-6}e^6$ &c.

3. Divide the Remainder by $mb^{m-1}+pb^{m-2}e+qb^{m-3}e^2+rb^{m-4}e^3$ &c. and the Quotient will be $+e$.

4. Add the Quotient $+e$ to the Root $+b$, and the Sum $b+e$ is the Root required. Q. E. E.

Whence we have the following universal Formula for extracting the Roots of all Powers in Numbers, when the Homologous Power b^m is assumed less than just.

$$
\begin{array}{rl}
 & b^m+mb^{m-1}e+pb^{m-2}e^2+qb^{m-3}e^3 \text{ \&c. } (b+e=r^{\frac{1}{m}} \\
\text{Subtract} & b^m \\
\hline
mb^{m-1}+pb^{m-2}e+qb^{m-3}e^2 \text{ \&c.}) & +mb^{m-1}e+pb^{m-2}e^2+qb^{m-3}e^3 \text{ \&c.} \\
 & +mb^{m-1}e+pb^{m-2}e^2+qb^{m-3}e^3 \text{ \&c.} \\
\hline
 & * \quad * \quad *
\end{array}
$$

From whence are deduced the following particular Formulæ;

(1st, For the Square Root.)

$$
\begin{array}{rl}
 & b^2+2be+e^2 \ (b+e=r^{\frac{1}{2}} \\
 & b^2 \\
\hline
2b+e) & 2be+e^2 \\
 & 2be+e^2 \\
\hline
 & * \quad *
\end{array}
$$

(2dly, For the Cube Root)

$$
\begin{array}{rl}
 & b^3+3b^2e+3be^2+e^3 \ (b+e=r^{\frac{1}{3}} \\
 & b^3 \\
\hline
3b^2+3be+e^2) & 3b^2e+3be^2+e^3 \\
 & 3b^2e+3be^2+e^3 \\
\hline
 & * \quad * \quad *
\end{array}
$$

 (3dly,

(*3dly*, For the Biquadrate Root.)

$$b^4+4b^3e+6b^2e^2+4be^3+e^4=r \quad (b+e=r^{\frac{1}{4}}$$
$$b^4$$
$$4b^3+6b^2e+4be^2+e^3) \quad 4b^3e+6b^2e^2+4be^3+e^4 \qquad \&c.$$
$$4b^3e+6b^2e^2+4be^3+e^4$$
$$* \quad * \quad * \quad *$$

Secondly, To extract the m Root from the Resolvend $b^m-mb^{m-1}e+pb^{m-2}e^2-qb^{m-3}e^3+rb^{m-4}e^4-sb^{m-5}e^5+tb^{m-6}e^6$ &c. $=r$.

Effection.

1. Find the Root b of the first Term b^m, setting it down as above.
2. Subtract the given Resolvend from b^m, and the Remainder will be $+mb^{m-1}e-pb^{m-2}e^2+qb^{m-3}e^3-rb^{m-4}e^4$ &c. (*In.* 384.)
3. Divide that Remainder by $-mb^{m-1}+pb^{m-2}e-qb^{m-3}e^2+rb^{m-4}e^3$ &c. and the Quotient will be $-e$. Therefore $b-e$ is the Root required. *Q. E. E.*

Whence we have this *universal Formula* for extracting the Roots of all Powers in Numbers, when b^m is assumed greater than just.

$$b^m \quad * \quad * \quad * \quad \&c.$$
$$\text{Subtract } b^m-mb^{m-1}e+pb^{m-2}e^2-qb^{m-3}e^3 \ \&c. \quad (b-e=r^{\frac{1}{m}}$$
$$-b^{m-1}+pb^{m-2}e-qb^{m-3}e^2 \ \&c.)+mb^{m-1}e-pb^{m-2}e^2+qb^{m-3}e^3 \ \&c.$$
$$mb^{m-1}e-pb^{m-2}e^2+qb^{m-3}e^3 \ \&c.$$
$$* \quad * \quad *$$

From whence are deduced these particular *Formulæ*.

(*1st*, For the Square Root.)

$$b^2 \quad * \quad *$$
$$b-2be+ee \quad (b-e=r^{\frac{1}{2}}$$
$$-2b+e.)+2be-ee$$
$$+2be-ee$$
$$* \quad *$$

(*2dly*, For the Cube Root.)

$$b^3 \quad * \quad * \quad *$$
$$b^3-3b^2e+3be^2-e^3 \quad (b-e=r^{\frac{1}{3}}$$
$$-3be^2+3be-e^2) \quad 3b^2e-3be^2+e^3$$
$$3b2e-3be^2+e^3$$
$$* \quad * \quad *$$

(*3dly*,

(3*dly*, For the Biquadrate Root.)

$$
\begin{array}{rl}
 & b^4 \quad * \quad\quad * \quad\quad * \quad\quad * \;. \\
 & b^4-4b^3e+6b^2e^2-4b^3e+e^4=r \quad (b-e=r^{\frac{1}{4}} \\
\hline
-4b^3e+6b^2e-4be^2+e^3) & 4b^3e-6b^2e^2+4b^3e-e^4 \\
 & 4b^3e-6b^2e^2+4b^3e-e^4 \\
\hline
 & * \quad\quad * \quad\quad * \quad\quad *
\end{array}
$$

SCHOLIUM III.

417. The following particular Examples are inferted for the Learner's Practice.

Example 1.

To extract the Square Root from $a^4+6a^3b+5a^2b^2-12ab^3+4b^4$.

$$
\begin{array}{lrl}
 & & r=a^4+6a^3b+5a^2b^2-12ab^3+4b^4 \quad (a^2+3ab-2bb=r^{\frac{1}{2}} \\
\text{Subtract} & & a^4 \\
 & & * \\
2a^2+3ab &) & 6a^3b+5a^2b^2 \\
 & & 6a^3b+9a^2b^2 \\
\hline
2a^2+6ab-2bb &) & -4a^2b^2-12ab^3+4b^4 \\
 & & -4a^2b^2-12ab^3+4b^4 \\
\hline
 & & * \quad\quad * \quad\quad *
\end{array}
$$

Example 2.

$$
\begin{array}{lrl}
 & & r=y^4+4y^3 \quad * \quad -8y+4 \quad (yy+2y-2=r^{\frac{1}{2}} \\
 & & y^4 \\
2yy+2y &) & * \quad 4y^3 \quad * \\
 & & 4y^3+4yy \\
\hline
2yy+4y &) & * \quad -4yy-8y+4 \\
 & & -4yy-8y+4 \\
\hline
 & & * \quad\quad * \quad\quad *
\end{array}
$$

Example

Example 3.

$$
\begin{array}{rl}
 & r=a^6+6a^5-40a^3+96a-64\ (a^2+2a+4=r^{\frac{1}{3}} \\
 & a^6 \\
\hline
3a^4+6a^3+4a^2)\ * & 6a^5-40a^3 \\
 & 6a^5+12a^4+8a^3 \\
\hline
3a^4+12a^3-24a+16)\ * & -12a^4-48a^3+96a-64 \\
 & -12a^4-48a^3+96a-64 \\
\hline
 & *\quad *\quad *\quad *
\end{array}
$$

Here, according to the foregoing Formula for the Cube Root above, $3b^2$ in the first Divisor $=3a^2$, $c=2a^2$, and $3bc=6a^3$: $3b^2$ in the second Divisor $=3\times\overline{a^2+2a}^2$, $c=-4$, and $3bc=3\times\overline{a^2+2a}\times-4$.

CHAP. IV.

Of the Arithmetic of Surds or Irrational Quantities.

Problem VII.

418. TO reduce *Surd Quantities* to their lowest Terms.

Effection.

A *Surd Quantity*, where the Root of the Whole cannot be extracted, is performed by extracting the Root of some Divisor of it. Thus $\overline{aabc}^{\frac{1}{2}}$ by extracting the Root of a^2 becomes $a|\overline{bc}^{\frac{1}{2}}$: And 48 by extracting the Root of the Divisor 16 becomes $4|\overline{3}^{\frac{1}{2}}$. And $\overline{48aabc}^{\frac{1}{2}}$, by extracting the Root of the Divisor $16aa$, becomes $4a|\overline{3bc}^{\frac{1}{2}}$. And $\overline{\frac{a^3b-4a^2b^2+4ab^3}{cc}}\Big|^{\frac{1}{2}}$, by extracting the Root of its Divisor $\frac{a^2-4ab+4bb}{cc}$ becomes $\frac{a-2b}{c}\times\overline{ab}^{\frac{1}{2}}$. And $\overline{\frac{a^2o^2m^2}{p^2z^2}+\frac{4a^2m^2}{pz^2}}\Big|^{\frac{1}{2}}$, by extracting the Root of the Divisor $\frac{a^2m^2}{p^2z^2}$, becomes

comes $\frac{am}{pz}\overline{|oo+4mp}|^{\frac{1}{2}}$. And $6\times\frac{75^{\frac{1}{2}}}{98^{\frac{1}{2}}}$, by extracting the Root of the Divisor $\frac{25}{49}$ becomes $\frac{30}{7}\times\overline{\Big|\frac{3}{2}}|^{\frac{1}{2}}$ or $\frac{30}{7}\times\overline{\Big|\frac{6}{4}}|^{\frac{1}{2}}$, and by yet extracting the Root of the Denominator, it becomes $\frac{15}{7}\times 6^{\frac{1}{2}}$. And so $a\overline{\Big|\frac{b}{a}}|^{\frac{1}{2}}$ or $a\overline{\Big|\frac{ab}{aa}}|^{\frac{1}{2}}$, by extracting the Root of the Denominator, becomes $\overline{ab}|^{\frac{1}{2}}$. And $\overline{8a^3b+16a^4}|^{\frac{1}{3}}$, by extracting the Cube Root of its Divisor $8a^3$, becomes $2a\overline{|b+2a}|^{\frac{1}{3}}$. And not unlike this $\overline{a^3x}|^{\frac{1}{4}}$, by extracting the Biquadrate Root of its Divisor aa, becomes $a^{\frac{1}{2}}\overline{|ax}|^{\frac{1}{4}}$, or by extracting the Biquadrate Root of the Divisor a^4, becomes $a\overline{\Big|\frac{x}{a}}|^{\frac{1}{4}}$. And so $\overline{a^7x^5}|^{\frac{1}{6}}$ is changed into $a\overline{|ax^5}|^{\frac{1}{6}}$, or into $ax\overline{\Big|\frac{a}{x}}|^{\frac{1}{6}}$, or into $\overline{ax}|^{\frac{1}{2}}\times\overline{aax}|^{\frac{1}{3}}$. *Vide* Sir *Isaac Newton's* Algebra, as translated by Mr. *Ralphson* and Mr. *Cunn*, pag. 49, 50.

Scholium IV.

419. This *Reduction* is not only of Use for abbreviating Radical Quantities, but also for their Addition and Subtraction, if they agree in their Roots, when they are reduced to the most simple Form; for then they may be added, which otherwise cannot. Thus the Root $\overline{48}|^{\frac{1}{2}}+\overline{75}|^{\frac{1}{2}}$ by Reduction becomes $4\overline{|3}|^{\frac{1}{2}}+5\overline{|3}|^{\frac{1}{2}}=9\overline{|3}|^{\frac{1}{2}}$. And $\overline{48}|^{\frac{1}{2}}-\overline{\frac{16}{27}}|^{\frac{1}{2}}$ by Reduction becomes $4\overline{|3}|^{\frac{1}{2}}-\frac{4}{9}\overline{|3}|^{\frac{1}{2}}=\frac{32}{9}\overline{|3}|^{\frac{1}{2}}$. And thus $\overline{\Big|\frac{4ab^3}{cc}}|^{\frac{1}{2}}+\overline{\Big|\frac{a^3b-4a^2b+4ab^3}{cc}}|^{\frac{1}{2}}$ by extracting what is rational in it, becomes $\frac{2b}{c}\overline{|ab}|^{\frac{1}{2}}+\frac{a-2b}{c}\overline{|ab}|^{\frac{1}{2}}=\frac{a}{c}\overline{|ab}|^{\frac{1}{2}}$. And $\overline{8a^3b+16a^4}|^{\frac{1}{3}}-\overline{b^4+2ab^3}|^{\frac{1}{3}}$ becomes $2a\overline{|b+2a}|^{\frac{1}{3}}-b\overline{|b+2a}|^{\frac{1}{3}}=\overline{2a-b}\times\overline{b+2a}|^{\frac{1}{3}}$. *Newton's Algebra*, p. 50. And in Numbers $8^{\frac{1}{2}}+18^{\frac{1}{2}}=2\overline{|2}|^{\frac{1}{2}}+$

$3\overline{2}^{\frac{1}{2}}=5\overline{2}^{\frac{1}{2}}=\overline{50}^{\frac{1}{2}}$. $24^{\frac{1}{3}}+81^{\frac{1}{3}}=2\overline{3}^{\frac{1}{3}}+3\overline{3}^{\frac{1}{3}}=5\overline{3}^{\frac{1}{3}}=\overline{375}^{\frac{1}{3}}$. $\overline{50}^{\frac{1}{2}}-\overline{18}^{\frac{1}{2}}=5\overline{2}^{\frac{1}{2}}-3\overline{2}^{\frac{1}{2}}=2\overline{2}^{\frac{1}{2}}=8^{\frac{1}{2}}$.

PROBLEM VIII.

420. To reduce *Surd Quantities* of different Denominations to the ſame Denomination.

Effection.

Reduce their fractional Exponents to the ſame Denomination (*In.* 107.) and involve each given Quantity to the Power, which the Numerator of its Exponent ſo reduced, ſhall direct. *Ex. gr.* $z^{\frac{n}{m}}$ and $y^{\frac{r}{s}}$ reduced to the ſame Denomination become $z^{\frac{ns}{ms}}$ and $y^{\frac{mr}{ms}}$. a^m and $b^{\frac{1}{2}}$ become $a^{\frac{2m}{2}}$ and $b^{\frac{1}{2}}$. $\overline{ax}^{\frac{1}{2}}$ and $\overline{aax}^{\frac{1}{3}}$ become $\overline{ax}^{\frac{3}{6}}$ and $\overline{aax}^{\frac{2}{6}}$, or $\overline{aaaxxx}^{\frac{1}{6}}$ and $\overline{aaaaxx}^{\frac{1}{6}}$. $a^{\frac{1}{2}}$ and $\overline{ab}^{\frac{1}{4}}$ become $a^{\frac{4}{8}}$ and $\overline{ab}^{\frac{2}{8}}$ or $\overline{aaaa}^{\frac{1}{8}}$ and $\overline{aabb}^{\frac{1}{8}}$; $8^{\frac{1}{2}}$ and $\overline{\frac{3}{8}}^{\frac{1}{5}}$ become $8^{\frac{5}{10}}$ and $\overline{\frac{3}{8}}^{\frac{2}{10}}$, or $\overline{32768}^{\frac{1}{10}}$, and $\overline{\frac{9}{64}}^{\frac{1}{10}}$, $6^{\frac{1}{2}}$ and $\overline{\frac{5}{6}}^{\frac{1}{4}}$ become $6^{\frac{4}{8}}$ and $\overline{\frac{5}{6}}^{\frac{2}{8}}$; or $\overline{1296}^{\frac{1}{8}}$ and $\overline{\frac{25}{6}}^{\frac{1}{8}}$. a and $\overline{bc}^{\frac{1}{2}}$ become $a^{\frac{2}{2}}$ and $\overline{bc}^{\frac{1}{2}}$ or $\overline{aa}^{\frac{1}{2}}$ and $\overline{bc}^{\frac{1}{2}}$; $2a$ and $\overline{b+2a}^{\frac{1}{3}}$ become $\overline{2a}^{\frac{3}{3}}$ and $\overline{b+2a}^{\frac{1}{3}}$ or $\overline{8aaa}^{\frac{1}{3}}$ and $\overline{b+2a}^{\frac{1}{3}}$; $6abb$ and $\overline{18ab^3}^{\frac{1}{2}}$ become $\overline{6abb}^{\frac{2}{2}}$ and $\overline{18ab^3}^{\frac{1}{2}}$, &c.

PROBLEM IX.

421. To multiply or divide one Homogeneous Surd by another.

Effection.

1. Reduce the Exponents to the ſame Denomination. (*In.* 107.)
2. Add or ſubtract theſe Exponents ſo reduced, according as they are to be multiplied or divided (*In.* 153.)
3. Set the Sum or Difference for the Exponent of the Product or Quotient, and it is done.

Examples

Examples in Multiplication.

$a^{\frac{n}{m}} \times a^{\frac{r}{s}} = a^{\frac{ns+rm}{ms}}$. $b^m \times b^{\frac{1}{2}} = b^{\frac{2m+1}{2}}$. $3z|\overline{b-c}^{\frac{1}{2}} \times 5a|\overline{b-c}^{\frac{1}{2}} = 15az|\overline{b-c}^{\frac{2}{2}} =$ $15azb - 15azc$. $2a|\overline{n}^{\frac{2}{3}} \times 3b|\overline{n}^{\frac{1}{2}} = 2a|\overline{n}^{\frac{4}{6}} \times 3b|\overline{n}^{\frac{3}{6}} = 6ab|\overline{n}^{\frac{7}{6}} = 6abn|\overline{n}^{\frac{1}{6}}$. $b^{\frac{2}{3}} \times b^{\frac{1}{3}}$ $= b^{\frac{3}{3}}$ or b. $z^{\frac{2}{4}} \times z^{\frac{1}{4}} = z^{\frac{3}{4}}$ or $z|\overline{z}^{\frac{1}{4}}$

Examples in Division.

$a^{\frac{ns+rm}{ms}} \div a^{\frac{n}{m}} = a^{\frac{ns+rm}{ms} - \frac{n}{m}} = a^{\frac{rm}{sm}}$ or $a^{\frac{r}{s}}$. $b^{\frac{2m-1}{2}} \div b^m = b^{\frac{1}{2}}$. $\overline{15azb-15azc}$ $\div 3z|\overline{b-c}^{\frac{1}{2}} = 5a|\overline{b-c}^{\frac{1}{2}}$. $6abn|\overline{n}^{\frac{1}{6}} \div 2a|\overline{n}^{\frac{2}{3}} = 3b|\overline{n}^{\frac{1}{2}}$. $b \div b^{\frac{1}{3}} = b^{\frac{2}{3}}$. $z|z^{\frac{2}{4}} \div$ $z^{\frac{2}{4}} = z^{\frac{1}{4}}$.

Problem X.

422. To multiply or divide one Heterogeneous Surd by another.

Effection.

1. Reduce them to the same Denomination (In. 420.)
2. Mutiply or divide the Quantities so reduced by each other, setting the common Exponent over the Product or Quotient, and it is done.

Examples in Multiplication.

$b^{\frac{2}{3}} \times c^{\frac{2}{3}} = \overline{bb}^{\frac{1}{3}} \times \overline{cc}^{\frac{1}{3}} = \overline{bbcc}^{\frac{1}{3}}$. $b^{\frac{3}{4}} \times c^{\frac{1}{2}} = b^{\frac{6}{8}} \times c^{\frac{4}{8}} = b^{\frac{3}{4}} \times c^{\frac{2}{4}} = \overline{bbb}^{\frac{1}{4}} \times \overline{cc}^{\frac{1}{4}} = |\overline{bbbcc}^{\frac{1}{4}}$. $5|\overline{b}^{\frac{2}{5}} \times 2a|\overline{c}^{\frac{3}{5}} = 5|\overline{bb}^{\frac{1}{5}} \times 2a|\overline{ccc}^{\frac{1}{5}} = 10a|\overline{bbccc}^{\frac{1}{5}}$. $\overline{ax}^{\frac{1}{2}} \times \overline{aax}^{\frac{1}{3}} = \overline{ax}^{\frac{3}{6}} \times \overline{aax}^{\frac{2}{6}} =$ $\overline{aaaxxx}^{\frac{1}{6}} \times \overline{aaaaxx}^{\frac{1}{6}} = \overline{a^7x^5}^{\frac{1}{6}}$. $6^{\frac{1}{2}} \times |\overline{\tfrac{1}{6}}^{\frac{1}{4}} = 6^{\frac{2}{4}} \times |\overline{\tfrac{1}{6}}^{\frac{1}{4}} = \overline{6 \times 6}^{\frac{1}{4}} \times |\overline{\tfrac{1}{6}}^{\frac{1}{4}} = |\overline{30}^{\frac{1}{4}}$. ax $\overline{c-x}^{\frac{1}{2}} = \overline{bb}^{\frac{1}{2}} \times \overline{c-x}^{\frac{1}{2}} = \overline{bbc-bbx}^{\frac{1}{2}}$.

$ax|\overline{bc}^{\frac{1}{2}} = a^{\frac{2}{2}} \times \overline{bc}^{\frac{1}{2}} = \overline{aabc}^{\frac{1}{2}}$. $4ax|\overline{3bc}^{\frac{1}{2}} = \overline{48a^2bc}^{\frac{1}{2}}$. $2ax\overline{b+2a}^{\frac{1}{3}} = \overline{8a^3b+16a^4}^{\frac{1}{3}}$.

$$
\begin{array}{l}
8^{\frac{2}{2}} + 2^{\frac{1}{2}} + \overline{32}^{\frac{1}{2}} \\
8^{\frac{1}{2}} + 2^{\frac{1}{2}} + 32^{\frac{1}{2}} \\
\hline
16 + 8 + 32 \\
4 + 2 + 8 \\
8 + 4 + 16 \\
\hline
8 + 8 + 34 + 16 + 32 = 98
\end{array}
$$

Examples

Examples in Division.

$\overline{bbcc}^{\frac{1}{3}} \div \overline{bb}^{\frac{1}{3}} = \overline{cc}^{\frac{1}{3}}$ $\overline{b^3c^2}^{\frac{1}{4}} \div c^{\frac{1}{2}} = \overline{bbb}^{\frac{1}{4}}$ or $b^{\frac{3}{4}}$. $10a|\overline{bbccc}^{\frac{1}{5}} \div 2a|\overline{ccc}^{\frac{1}{5}} = 5|\overline{bb}^{\frac{1}{5}}$ or $5|b^{\frac{2}{5}}$. $\overline{a^7x^5}^{\frac{1}{6}} \div \overline{aax}^{\frac{1}{3}} = \overline{ax}^{\frac{1}{2}}$. $|\overline{30}^{\frac{1}{4}} \div 6^{\frac{1}{2}} = |\overline{\tfrac{5}{6}}^{\frac{1}{4}}$. $\overline{a^2bc}^{\frac{1}{2}} \div |\overline{bc}^{\frac{1}{2}} = a$. $\overline{48a^2bc}^{\frac{1}{2}} \div 4a = |\overline{3bc}^{\frac{1}{2}}$. $\overline{8a^3b+16a^4}^{\frac{1}{3}} \div \overline{b+2a}^{\frac{1}{3}} = 2a$. $\overline{ac}^{\frac{1}{2}} \div b = |\overline{\tfrac{ac}{bb}}^{\frac{1}{2}}$. $6abb \div 18ab^3$

$= |\overline{\tfrac{36aab^4}{18abbb}}^{\frac{1}{2}} = |\overline{2ab}^{\frac{1}{2}}$

$$
\begin{array}{l}
3^{\frac{1}{2}})\ \overline{15}^{\frac{1}{2}} - 6^{\frac{1}{2}} + \overline{12}^{\frac{1}{2}}\ (5^{\frac{1}{2}} - 2^{\frac{1}{2}} + 2 \\
\quad\ \ \underline{\overline{15}^{\frac{1}{2}}} \\
\quad\ \ \ * - 6^{\frac{1}{2}} \\
\quad\ \ \ \ \underline{- 6^{\frac{1}{2}}} \\
\quad\ \ \ \ \ \ \ * + 12^{\frac{1}{2}} \\
\quad\ \ \ \ \ \ \ \ \ \underline{+ 12} \\
\quad\ \ \ \ \ \ \ \ \ \ \ *
\end{array}
$$

Examples in Multiplication, when either or both the Factors are in this Form, $\overline{a+b^{\frac{1}{2}}}^{\frac{1}{2}}$ *called* Univerfal Surds.

$\overline{3+2^{\frac{1}{2}}} \times |\overline{2^{\frac{1}{2}}}^{\frac{1}{2}} = \overline{3+2^{\frac{1}{2}}} \times 2^{\frac{1}{4}} = 3|\overline{2}^{\frac{1}{4}} + 2^{\frac{3}{4}}$ or $\overline{162}^{\frac{1}{4}} + 8^{\frac{1}{4}}$. $\overline{4+3^{\frac{1}{2}}}^{\frac{1}{2}} \times \overline{2+5^{\frac{1}{2}}}^{\frac{1}{2}} = \overline{\overline{4+3^{\frac{1}{2}}} \times \overline{2+5^{\frac{1}{2}}}}^{\frac{1}{2}} = \overline{8+2|\overline{3}^{\frac{1}{2}}+4|\overline{5}^{\frac{1}{2}}+15^{\frac{1}{2}}}^{\frac{1}{2}}$. $\overline{4+3^{\frac{1}{2}}}^{\frac{1}{2}} \times \overline{4-3^{\frac{1}{2}}}^{\frac{1}{2}} = \overline{\overline{4+3^{\frac{1}{2}}} \times \overline{4-3^{\frac{1}{2}}}}^{\frac{1}{2}} = \overline{16+4|\overline{3}^{\frac{1}{2}}-4|\overline{3}^{\frac{1}{2}}-3}^{\frac{1}{2}} = |\overline{13}^{\frac{1}{2}}$. $\overline{3+2^{\frac{1}{2}}}^{\frac{1}{3}} \times \overline{3-2^{\frac{1}{3}}} = \overline{3+2^{\frac{1}{2}}}^{\frac{1}{3}} \times \overline{3-2^{\frac{1}{3}}}^{\frac{2}{3}} = \overline{3+2^{\frac{1}{2}}}^{\frac{1}{3}} \times \overline{25-27|\overline{2}^{\frac{1}{3}}+9|4^{\frac{1}{3}}}^{\frac{1}{3}}$

$= \overline{75-81|\overline{2}^{\frac{1}{3}}+27|\overline{4}^{\frac{1}{3}}+25|\overline{2}^{\frac{1}{2}}-27|\overline{32}^{\frac{1}{6}}+9|\overline{128}^{\frac{1}{6}}}^{\frac{1}{3}}$.

Examples

Examples in Division.

$$\overline{162+8}^{\frac{1}{4}} \div 3+2^{\frac{1}{2}} = 2^{\frac{1}{4}}. \quad \overline{8+2|\overline{3}^{\frac{1}{2}}+4|\overline{5}^{\frac{1}{2}}+15^{\frac{1}{2}}}^{\frac{1}{2}} \div \overline{4+3^{\frac{1}{2}}}^{\frac{1}{2}} = \overline{2+5^{\frac{1}{2}}}^{\frac{1}{2}}$$

$$\overline{4+3^{\frac{1}{2}}}^{\frac{1}{2}}\Big)\,\overline{16-3}^{\frac{1}{2}}\Big(\overline{4-3^{\frac{1}{2}}}^{\frac{1}{2}}$$

$$\underline{16+4|\overline{3}^{\frac{1}{2}}}$$

$$-4|\overline{3}^{\frac{1}{2}}-3$$

$$\underline{-4|\overline{3}^{\frac{1}{2}}-3}$$

$$0 \quad 0$$

$$\overline{5+2^{\frac{1}{2}}}^{\frac{1}{3}}\Big)\,\overline{75+25|\overline{2}^{\frac{1}{2}}-81|\overline{2}^{\frac{1}{3}}-27|\overline{2}^{\frac{5}{6}}+27|\overline{4}^{\frac{1}{3}}+9|\overline{128}^{\frac{1}{6}}}^{\frac{1}{3}}$$

$$\underline{75+25|\overline{2}^{\frac{1}{2}}} \qquad \Big(25-27|\overline{2}^{\frac{1}{3}}+9|\overline{4}^{\frac{1}{3}}$$

$$0 \quad 0-81|\overline{2}^{\frac{1}{3}}-27|\overline{2}^{\frac{5}{6}}$$

$$\underline{-81|\overline{2}^{\frac{1}{3}}-27|\overline{2}^{\frac{5}{6}}}$$

$$0 \quad 0 \qquad +27|\overline{4}^{\frac{1}{3}}+9|\overline{128}^{\frac{1}{6}}$$

$$\underline{27|\overline{4}^{\frac{1}{3}}+9|\overline{128}^{\frac{1}{6}}}$$

$$0 \quad 0$$

SCHOLIUM V.

423. The *Addition* and *Subtraction* of *impossible Roots* is all one with that of *real* ones (In. 419.) But the Mulplication and Division of the same is different. For every Defective Quantity under its radical Sign is considered as Positive; and therefore the Products and Quotients made by such, do always keep the same Sign; since otherwise a Positive or real Product would be raised from impossible Factors, which would be absurd.

Examples in Addition and Subtraction.

To $\overline{|-27}^{\frac{1}{2}} = 3\overline{|-3}^{\frac{1}{2}}$ $\qquad$ $\overline{|-32}^{\frac{1}{2}} = 2\overline{|-2}^{\frac{1}{4}}$

Add $\overline{|-12}^{\frac{1}{2}} = 2\overline{|-3}^{\frac{1}{2}}$ $\qquad$ $\overline{|-162}^{\frac{1}{3}} = 3\overline{|-2}^{\frac{1}{4}}$

Sum $\overline{|-27}^{\frac{1}{2}} + \overline{|-3}^{\frac{1}{2}} = 5\overline{|-3}^{\frac{1}{2}}$ $\qquad$ $\overline{|-32}^{\frac{1}{4}} + \overline{|-162}^{\frac{1}{4}} = 5\overline{|-24}^{\frac{1}{4}}$

From $\overline{|-27}^{\frac{1}{2}} = 3\overline{-3}^{\frac{1}{2}}$

Subtract $\overline{|-12}^{\frac{1}{2}} = 2\overline{|-3}^{\frac{1}{2}}$

Remains $\overline{|-27}^{\frac{1}{2}} - \overline{|-12}^{\frac{1}{2}} = \overline{|-3}^{\frac{1}{2}}$

Examples in Multiplication.

$$\overline{|-6}^{\frac{1}{2}} + \overline{|-2}^{\frac{1}{2}} - \overline{|-5}^{\frac{1}{2}}$$

$$+\overline{|-3}^{\frac{1}{2}}$$

$$+\overline{|-18}^{\frac{1}{2}} + \overline{|-6}^{\frac{1}{2}} - \overline{|-15}^{\frac{1}{2}}$$

$$5\overline{|-3}^{\frac{1}{2}} + 3\overline{|-2}^{\frac{1}{2}} - 2\overline{|-5}^{\frac{1}{2}} + 2$$

$$3\overline{|-5}^{\frac{1}{2}} - 2\overline{|-3}^{\frac{1}{2}}$$

$$-10\overline{|-9}^{\frac{1}{2}} - 6\overline{|-6}^{\frac{1}{2}} + 4\overline{|-15}^{\frac{1}{2}} - 4\overline{|-3}^{\frac{1}{2}}$$

$$15\overline{|-15}^{\frac{1}{2}} + 9\overline{|-10}^{\frac{1}{2}} - 6\overline{|-25}^{\frac{1}{2}} + 6\overline{|-5}^{\frac{1}{2}}$$

$$19\overline{|-15}^{\frac{1}{2}} + 9\overline{|-10}^{\frac{1}{2}} - 10\overline{|-9}^{\frac{1}{2}} - 6\overline{|-25}^{\frac{1}{2}} - 6\overline{|-6}^{\frac{1}{2}} + 6\overline{|-5}^{\frac{1}{2}} - 4\overline{|-3}^{\frac{1}{2}}$$

Examples in Division.

$$+\overline{|-3}^{\frac{1}{2}}\Big)\ \overline{|-18}^{\frac{1}{2}} + \overline{|-6}^{\frac{1}{2}} - \overline{|-15}^{\frac{1}{2}}\ \Big(\overline{|-6}^{\frac{1}{2}} + \overline{|-2}^{\frac{1}{2}} - \overline{|-5}^{\frac{1}{2}}$$

$$\overline{|-18}^{\frac{1}{2}}$$

$$0 \qquad \overline{|-6}^{\frac{1}{2}}$$

$$\overline{|-6}^{\frac{1}{2}}$$

$$0 \qquad -\overline{|-15}^{\frac{1}{2}}$$

$$-\overline{|-15}^{\frac{1}{2}}$$

$$0$$

$$3|\overline{-5}^{\frac{1}{2}}-2|\overline{-3}^{\frac{1}{2}})\,15|\overline{-15}^{\frac{1}{2}}-10|\overline{-9}^{\frac{1}{2}}+9|\overline{-10}^{\frac{1}{2}}-6|\overline{-6}^{\frac{1}{2}}-6|\overline{-25}^{\frac{1}{2}}+4|\overline{-15}^{\frac{1}{2}}+6|\overline{-5}^{\frac{1}{2}}-4|\overline{-3}^{\frac{1}{2}}\left(\begin{array}{l}5|\overline{-3}^{\frac{1}{2}}+\\3|\overline{-2}^{\frac{1}{2}}-2|\overline{-5}^{\frac{1}{2}}+2\end{array}\right.$$

$$15|\overline{-15}^{\frac{1}{2}}-10|\overline{-9}^{\frac{1}{2}}$$

$$*\quad *\quad 9|\overline{-10}^{\frac{1}{2}}-6|\overline{-6}^{\frac{1}{2}}$$

$$9|\overline{-10}^{\frac{1}{2}}-6|\overline{-6}^{\frac{1}{2}}$$

$$*\quad *\quad -6|\overline{25}^{\frac{1}{2}}+4|\overline{-15}^{\frac{1}{2}}$$

$$-6|\overline{25}^{\frac{1}{2}}+4|\overline{-15}^{\frac{1}{2}}$$

$$*\quad *\quad +6|\overline{-5}^{\frac{1}{2}}-4|\overline{-3}^{\frac{1}{2}}$$

$$+6|\overline{-5}^{\frac{1}{2}}-4|\overline{-3}^{\frac{1}{2}}$$

$$*\quad *$$

CHAP.

CHAP. V.

Of the Invention of Divisors.

424. TO find the Divisors of Quantity where it admits of Division.

The following *Effection* is taken from Sir *Isaac Newton*'s Algebra, pag. 38, *&c.*

If the Quantity be simple, divide it by its least Divisor, and the Quotient by its least Divisor, till there remain an indivisible Quotient, and you will have all the prime Divisors of [that] Quantity. Then multiply together each Pair of these Divisors, each Ternary [or three] of them, each Quaternary, *&c.* and you will also have all the compounded Divisors. As, if all the Divisors of the Number 60 are required, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and there will remain the indivisible Quotient 5. Therefore the prime Divisors are 1, 2, 2, 3, 5; those composed of the Pairs 4, 6, 10, 15; of the Ternaries 12, 20, 30; and of all of them 60. Again, If all the Divisors of the Quantity $21abb$ are desired, divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and there will remain the prime Quotient b. Therefore the prime Divisors are 1, 3, 7, a, b, b; and those composed of the Pairs 21, $3a$, $3b$, $7a$, $7b$, ab, bb; those composed of the Ternaries $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; and those of the Quaternaries $21ab$, $21bb$, $3abb$, $7abb$; that of the Quinaries $21abb$. After the same Way all the Divisors of $2abb - 6aac$ are 1, 2, a, $bb - 3ac$, $2a$, $2bb - 6ac$, $abb - 3aac$, $2abb - 6aac$.

If after a Quantity is divided by all its simple Divisors, it remains [still] compounded, and you suspect it has some compounded Divisor, [order it or] dispose it according to the Dimensions of any of the Letters in it; and in the Room of that Letter substitute successively three or more Terms of this Arithmetical Progression, *viz.* 3, 2, 1, 0, —1, —2, and set the resulting Terms together with all their Divisors, by the corresponding Terms of the Progression, setting down also the Signs of the Divisors, both Affirmative and Negative. Then set also down the Arithmetical Progressions which run through the Divisors of all the Numbers proceeding from the greater Terms to the less, in the Order that the Terms of the Progression 3, 2, 1, 0, —1, —2, proceed, and whose Terms differ either by Unity, or by some Number which divides the highest Term of the Quantity proposed.

posed. If any Progression of this kind occurs, that Term of it which stands in the same Line with the Term o of the first Progression, divided by the Difference of the Terms, will compose the Quantity by which you are to attempt the Division.

As if the Quantity be $x^3 - xx - 10x + 6$, by substituting, one by one, the Terms of this Progression 1. o. —1, for x, there will arise the Numbers —4, 6, + 14, which, together with all their Divisors, I place right against the Terms of the Progression 1. o. —1. after this Manner:

1	4	1. 2. 4.	+ 4.
o	6	1. 2. 3. 6.	+ 3.
—1	14	1. 2. 7. 14.	+ 2.

Then, because the highest Term x^3 is divisible by no Number but Unity, I seek among the Divisors a Progression whose Terms differ by Unity, and (proceeding from the highest to the lowest) decrease as the Terms of the lateral Progression 1. o. —1. And I find only one Progression of this Sort, *viz.* 4. 3. 2. whose Term therefore + 3 I chuse, which stands in the same Line with the Term o of the first Progression 1. o. —1. and I attempt the Division by $x - 3$, and [find] it succeeds, there coming out $xx - 4x + 2$.

Again, if the Quantity be $6y^4 - y^3 - 21yy + 3y + 20$, for y I substitute successively 1. o. —1. and the resulting Numbers 7. 20. 9. with all their Divisors, I place by them as follows:

1	7	1. 7.	7.
o	20	1. 2. 4. 5. 10. 20.	4.
—1	9	1, 3. 9	1.

And among the Divisors I perceive there is this decreasing Arithmetical Progression 7. 4. 1. The Difference of the Terms of this Progression, *viz.* 3 divides the highest Term of the Quantity $6y^4$. Wherefore I adjoin the Term + 4, which stands [in the Row] opposite to the Term o, divided by the Difference of the Terms, *viz.* 3. and I attempt the Division by $y + \frac{4}{3}$, or, which is the same Thing, by $3y + 4$, and the Business succeeds, there coming out $2^3y - 3yy - 3y + 5$.

And so, if the Quantity be $24a^5 - 50a^4 + 49a^3 - 140a^2 + 64a + 30$, the Operation will be as follows:

2	42	1.2.3.6.7.14.21.42	+3. +3. + 7.
1	23	1.23.	+1. —1. + 1.
o	30	1.2.3.5.6.10.15.30	—1. —5. — 5.
—1	297	1.3.9.11.27.33.99.297	—3. —9. —11.

 Here

Here are three Progressions, whose Terms —1.—5.—5. divided by the Differences of the Terms 2, 4, 6, give three Divisors to be try'd $a-\frac{1}{2}$, $a-\frac{1}{4}$, and $a-\frac{5}{6}$. And the Division by the last Divisor $a-\frac{5}{6}$, or $6a-5$, succeeds, there coming out $4a^4-5a^3+4aa-20a-6$.

If no Divisor occur by this Method, or none that divides the Quantity propos'd, we are to conclude, that that Quantity does not admit a Divisor of one Dimension. But perhaps it may, if it be a Quantity of more than three Dimensions, admit a Divisor of two Dimensions. And if so, that Divisor will be found by this Method. Substitute in that Quantity for the Letter [or Species] as before, four or more Terms of this Progression 3, 2, 1. 0.—1.—2.—3. Add and subtract singly all the Divisors of the Numbers that result to or from the Squares of the correspondent Terms of that Progression, multiply'd into some Numeral Divisor of the highest Term of the Quantity propos'd, and place right against the Progression the Sums and Differences. Then note all the collateral Progressions which run through those Sums and Difference. Then suppose $\mp$ C to be a Term of such a prime Progression, and $\mp$ B the Difference which arises by subducting $\mp$ C from the next superior Term which stands against the Term 1 of the first Progression, and A to be the aforesaid Numeral Divisor of the highest Term, and l [to be] a Letter which is in the propos'd Quantity, then $All \pm Bl \pm C$ will be the Divisor to be try'd.

Thus suppose the propos'd Quantity to be $x^4-x^3-5xx+12x-6$, for x I write successively 3, 2, 1, 0,—1, and the Numbers that come out 39.6.1.—6.—21.—26. I dispose [or place] together with their Divisors in another Column in the same Line with them, and I add and subtract the Divisors to and from the Squares of the Terms of the first Progression, multiply'd by the Numeral Divisor of the Term x^4, which is Unity, *viz.* to and from the Terms 9.4.1.0.1.4, and I dispose likewise the Sums and Differences on the Side. Then I write, as follows, the Progressions which occur among the same. Then I make use of the Terms of these Progressions 2 and —3, which stand opposite to the Term 0 in that Progression, which is in the first Column, successively

3	39	1.3.13.39	9	—30.—4.6.8.10.12.22.48	—4.	6
2	6	1.2. 3. 6	4	—2.1.2.3.5.6.7.10.	—2.	3
1	1	1.	1	0. 2.	0.	0
0	6	1.2. 3. 6	0	—6.—3.—2.—1.1.2.3.6	2	—3
—1	21	1.3. 7.21	1	—20.—6.—2.0.2.4.8.22	4	—6
—2	26	1.2.13.26	4	—22.—9.2.3.5.6.17.30	6	—9

for $\mp$ C, and I make use of the Differences that arise by subtracting these Terms from the superior Terms 0 and 0, *viz.* —2 and +3 respectively for

$\mp$ B.

$\mp$ B. Alſo Unity for A; and x for l. And ſo in the Room of $All \pm Bl \pm C$, I have theſe two Diviſors to try, *viz.* $xx+2x-2$, and $xx-3x+3$, by both of which the Buſineſs ſucceeds.

Again, if the Quantity $3y^5-6y^4+y^3-8yy-14y+14$ be propoſed, the Operation will be as follows. Firſt, I attempt the Buſineſs by adding and ſubtracting to and from the Squares of the Terms of the Progreſſion 1.0. —1, making uſe of 1 firſt, but the Buſineſs does not ſucceed.

3	170		27		—7. 17
2	38	1.2.19.38	12	—16—7.10.11.13.14.31.50	—7.—11
1	10	1.2. 5.10	3	—7.—2.1.2.4.5.8.13	—7. 5
0	14	1.2. 7.14	0	—14—7.—2.—1.1.2.7.14	—7.— 1
—1	10	1.2. 5.10	3	—7.—2.1.2.4.5.8.13	—7.— 7
—2	190		12		—7.—13

Wherefore, in the Room of A, I make uſe of 3, the other Diviſor of the higheſt Term; and theſe Squares being multiply'd by 3, I add and ſubtract the Diviſors to and from the Products, *viz.* 12.3.0.3. and I find theſe two Progreſſions in the reſulting Terms, —7.—7.—7.—7, and 11.5.—1.—7. For Expedition ſake, I had neglected the Diviſors of the outermoſt Terms 170 and 190. Wherefore, the Progreſſions being continued upwards and downwards, I take the next Terms, *viz.* —7 and 17 at the Top, and —7 and —13 at Bottom, and I try if theſe being ſubducted from the Numbers 27 and 12, which ſtand againſt them in the 4th Column, [their] Differences divide thoſe [Numbers] 170 and 190, which ſtand againſt them in the ſecond Column. And the Difference between 27 and —7, that is, 34, divides 170; and the Difference of 12 and —7, that is, 19, divides 190. Alſo the Difference between 27 and 17, that is, 10, divides 170, but the Difference between 12 and —13, that is, 25, does not divide 190. Wherefore I reject the latter Progreſſion. According to the former, $\mp$ C is —7, and $\mp$ B is nothing; the Terms of the Progreſſion having no Difference. Wherefore the Diviſor to be try'd $All \pm Bl \pm C$ will be $3yy+7$. And the Diviſion ſucceeds, there coming out $y^3-2yy-2y+2$.

If after this Way, there can be found no Diviſor which ſucceeds, we are to conclude, that the propos'd Quantity will not admit of a Diviſor of two Dimenſions. The ſame Method may be extended to the Invention of Diviſors of more Dimenſions, by ſeeking in the aforeſaid Terms and Differences, not Arithmetical Progreſſions, but ſome others, the firſt, ſecond, and third Differences of whoſe Terms are in Arithmetical Progreſſion: But the Learner ought not to be detain'd about them.

Where there are two Letters in the propos'd Quantity, and all its Terms aſcend to equally high Dimenſions; put Unity for one of thoſe Letters, then

then, by the preceding Rules, seek a Divisor, and compleat the deficient Dimensions of this Divisor, by restoring that Letter for Unity. As if the Quantity be $6y^4-cy^3-21ccyy+3c^3y+20c^4$, where all the Terms are of four Dimensions, for c I put 1, and the Quantity becomes $6y^4-y^3-21yy+3y+20$, whose Divisor, as above, is $3y+4$; and having compleated the deficient Dimension of the last Term by a [correspondent] Dimension of c, you have $3y+4c$ [for] the Divisor sought. So, if the Quantity be $x^4-bx^3-5bbxx+12b^3x-6b^4$, putting 1 for b, and having found $xx+2x-2$ the Divisor of the resulting Quantity $x^4-x^3-5xx+12x-6$, I compleat its deficient Dimensions by [respective] Dimensions of b, and so I have $xx+2bx+2bb$ the Divisor sought.

Where there are three or more Letters in the Quantity propos'd, and all its Terms ascend to the same Dimensions, the Divisor may be found by the precedent Rules; but more expeditiously after this Way: Seek all the Divisors of all the Terms in which some [one] of the Letters is not, and also of all the Terms in which some other of the Letters is not; as also of all the Terms in which a third, fourth and fifth Letter is not, if there are so many Letters; and so run over all the Letters: And in the same Line with those Letters place the Divisors respectively. Then see if in any Series of Divisors going through all the Letters, all the Parts involving, only one Letter can be as often found as there are Letters (excepting only one) in the Quantity propos'd; and [likewise] if the Parts involving two Letters [may be found] as often as there are Letters (excepting two) in the Quantity propos'd. If so, all those Parts taken together under their [proper] Signs will be the Divisor sought.

As if there were propos'd the Quantity $12x^3-14bxx+9cxx-12bbx-6bcx+8ccx+8b^3-12bbc-4bcc+6c^3$; the Divisors of one Dimension of the Terms $8b^3-12bbc-4bcc+6c^3$; in which x is not found out (by the preceding Rules) will be $2b-3c$, and $4b-6c$; and of the Terms $12x^3+9cxx+8ccx+6c^3$, in which b is not, there will be only one Divisor $4x+3c$; and of the Terms $12x^3-14bxx-12bbx+8b^3$, in which there is not c, there will be the Divisors $2x-b$ and $4x-2b$. I dispose these Divisors in the same Lines with the Letters x, b, c, as you here see;

$$\begin{array}{c|l} x & 2b-3c.\ 4b-6c. \\ b & 4x+3c. \\ c & 2x-b.\ 4x-2b. \end{array}$$

Since there are three Letters, and each of the Parts of the Divisors only involve one of the Letters, those Parts ought to be found twice in the Series of Divisors. But the Parts $4b$, $6c$, $2x$, b of the Divisors $4b-6c$ and $2x-b$, only occur once, and are not found any where out of those Divisors whereof they

they are Parts. Wherefore I neglect those Divisors. There remain only three Divisors $2b-3c$, $4x+3c$, and $4x-2b$. These are in the Series going through all the Letters x, b, c, and each of the Parts $2b$, $3c$, $4x$, are found in them twice as ought to be, and that with the same Signs, if only the Signs of the Divisor $2b-3c$ be changed, and in its place you write $-2b+3c$. For you may change the Signs of any Divisor. I take therefore all the Parts of these, *viz.* $2b$, $3c$, $4x$ once [apiece] under their [proper] Signs, and the Aggregate $-2b+3c+4x$ will be the Divisor which was to be found. For if by this you divide the propos'd Quantity, there will come out $3xx-2bx+2cc-4bb$.

Again, if the Quantity be $12x^5-10ax^4-9bx^4-26a^2x^3+12abx^3+6bbx^3+24a^3xx-8aabxx-8abbxx-24b^3xx-4a^3bx+6aabbx-12ab^3x+18b^4x+12a^4b+32aab^3-12b^5$, I place the Divisors of the Terms in which x is not, by x; and those Terms in which a is not, by a; and those in which b is not, by b, as you here see. Then I perceive that all those that are but of one Dimension are

$$\begin{array}{c|l} x & b.\ 2b.\ 4b.\ aa+3bb.\ 2aa+6bb.\ 4aa+12bb. \\ & bb-3aa.\ 2bb-6aa.\ 4bb-12aa. \\ a & 4xx-3bx+2bb.\ 12xx-2bx+6bb. \\ b & x.\ 2x.\ 3x-4a.\ 6x-8a.\ 3xx-4ax.\ 6xx-8ax. \\ & 2xx+ax-3aa.\ 4xx+2ax-6aa. \end{array}$$

to be rejected, because the Simple ones, $b.\ 2b.\ 4b.\ x.\ 2x$, and the Parts of the compounded ones, $3x-4a.\ 6x-8a$, are found but once in all the Divisors; but there are three Letters in the propos'd Quantity, and those Parts involve but one, and so ought to be found twice. In like Manner, the Divisors of two Dimensions, $aa+3bb.\ 2aa+6bb.\ 4aa+12bb.\ bb-3aa.$ and $4bb-12aa$ I reject, because their Parts $aa.\ 2aa.\ 4aa.\ bb.$ and $4bb.$ involving only one Letter a or b, are not found more than once. But the Parts $2bb$ and $6aa$ of the Divisor $2bb-6aa$, which is the only remaining one in the Line with x, and which likewise involve only one Letter, are found again [or twice], *viz.* the Part $2bb$ in the Divisor $4xx-3bx+2bb$, and the Part $6aa$ in the Divisor $4xx+2ax-6aa$. Moreover, these three Divisors are in a Series standing in the same Lines with the three Letters x, a, b; and all their Parts $2bb$, $6aa$, $4xx$, which involve only one Letter, are found twice in them, and that under their proper Signs; but the Parts $3bx$, $2ax$, which involve two Letters, occur but once in them. Wherefore, all the divers Parts of these three Divisors, $2bb$, $6aa$, $4xx$, $3bx$, $2ax$, connected under their proper Signs, will make the Divisors sought, *viz.* $2bb-6aa+4xx-3bx+2ax$. I therefore divide the Quantity propos'd by this [Divisor] and there arises $3x^3-4axx-2aab-6b^3$.

If

If all the Terms of any Quantity are not equally high, the deficient Dimensions must be filled up by the Dimensions of any assumed Letter; then having found a Divisor by the precedent Rules, the assumed Letter is to be blotted out. As if the Quantity be $12x^3-14bxx+9xx-12bbx-6bx+8x+8b^3-12b^2-4b+6$; assume any Letter, as c, and fill up the Dimensions of the Quantity propos'd by its Dimensions, after this Manner, $12x^3-14bxx+9cxx-12bbx-6bcx+8ccx+8b^3-12bbc-4bcc+6c^3$. Then having found out its Divisor $4x-2b+3c$, blot out c, and you'll have the Divisor required, *viz.* $4x-2b+3$.

Sometimes Divisors may be found more easily than by these Rules. As if some Letter in the propos'd Quantity be of only one Dimension, you may seek for the greatest common Divisor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found; for that Divisor will divide the whole. And if there is no such common Divisor, there will be no Divisor of the whole. For example: If there be propos'd the Quantity $x^4-3ax^3-8aaxx+18a^3x-cx^3+acxx+8aacx-6a^3c-8a^4$, let there be sought the common Divisor of the Terms $-cx^3+acxx+8aacx-6a^3c$, in which c is only of one Dimension, and of the remaining Terms $x^4-3ax^3-8aaxx+18a^3x-8a^4$, and that Divisor, *viz.* $xx+2ax-2aa$, will divide the whole Quantity.

But the greatest common Divisor of two Numbers, if it is not known [or does not appear] at first Sight, is found by a perpetual Subtraction of the less from the greater, and of the Remainder from the [last Quantity] subtracted; and that will be the sought Divisor, which leaves nothing. Thus, to find the greatest common Divisor of the Numbers 203 and 667, subtract thrice 203 from 667, and the Remainder 58 thrice from 203, and the Remainder 29 twice from 58, and there will remain nothing; which shews, that 29 is the Divisor sought.

After the same Mannnr the common Divisor in Species, when it is compounded, is found, by subtracting either Quantity, or its Multiple, from the other; if those Quantities and the Remainder be ordered [or ranged] according to the Dimensions of any Letter, as is shewn in Division, and be each Time managed by dividing them by all their Divisors, which are either Simple, or divide each of its Terms as if it were a Simple one. Thus, to find the greatest common Divisor of the Numerator and Denominator of this Fraction $\frac{x^4-3ax^3-8aaxx+18a^3x-8a^4}{x^3-axx-8aax+6a^3}$, multiply the Denominator by x, that its first Term may become the same with the first Term of the Numerator. Then subtract it, and there will remain $-2ax^3+12a^3x-8a^4$, which, being rightly ordered by dividing by $-2a$, becomes $x^3-6ax^2+4a^3$. Subtract this from the Denominator, and there will remain $-axx-2aax+2a^3$; which again divided by $-a$ becomes $xx+2ax-2aa$. Multiply this by x, that its first

first Term may become the same with the first Term of the last subtracted Quantity $x^3-6aax+4a^3$, from which it is to be [likewise] subtracted, and there will remain $-2axx-4aax+4a^3$, which divided by $-2a$, becomes also $xx+2ax-2aa$. And since this is the same with the former Remainder, and consequently being subtracted from it, will leave nothing, it will be the Divisor sought; by which the propos'd Fraction, by dividing both the Numerator and Denominator by it, may be reduced to a more Simple one, *viz.* to $\frac{xx-5ax+4aa}{x-3a}$.

And so, if you have the Fraction

$$\frac{6a^5+15a^4b-4a^3cc-10aabcc}{9a^3b-27aabc-6abcc+18bc^3}$$

its Terms must be first abbreviated, by dividing the Numerator by aa, and the Denominator by $3b$: Then subtracting twice $3a^3-9aac-2acc+6c^3$ from $6a^3+15aab-4acc-10bcc$, there will remain $\begin{matrix}+15b\\+18c\end{matrix}\,aa\,\begin{matrix}-10bcc.\\-12c^3\end{matrix}$ Which being ordered, by dividing each Term by $5b+6c$ after the same Way as if $5b+6c$ was a simple Quantity, it becomes $3aa-2cc$. This being multiply'd by a, subtract it from $3a^3-9aac-2acc+6c^3$, and there will remain $-9aac+6c^3$: which being again ordered by a Division by $-3c$, becomes also $3aa-2cc$, as before. Wherefore $3aa-2cc$ is the Divisor sought. Which being found, divide by it the Parts of the propos'd Fraction, and you'll have $\frac{2a^3+5aab}{3ab-9bc}$.

Now, if a common Divisor cannot be found after this Way, it is certain there is none at all; unless, perhaps, it be one of the Terms that abbreviate the Numerator and Denominator of the Fraction: As, if you have the Fraction $\frac{aadd-ccdd-aacc+c^4}{4aad-4acd-2acc+2c^3}$, and so dispose its Terms, according to the Dimensions of the d, that the Numerator may become $\begin{matrix}aa\\-cc\end{matrix}\,dd\,\begin{matrix}-aacc\\+c^4\end{matrix}$, and the Denominator $\begin{matrix}4aa\\-4ac\end{matrix}\,d\,\begin{matrix}-2acc\\+2c^3\end{matrix}$. This must first be abbreviated, by dividing each Term of the Numerator by $aa-cc$, and each of the Denominator by $2a-2c$, just as if $aa-cc$ and $2a-2c$ were simple Quantities; and so, in Room of the Numerator there will come out $dd-cc$, and in Room of the the Denominator $2ad-cc$, from which, thus prepared, no common Divisor can be obtained. But, out of the Terms $aa-cc$ and $2a-2c$, by which both the Numerator and Denominator are abbreviated, there comes out a Divisor, *viz.* $a-c$, by which the Fraction may be reduced to this, *viz.*

add+

$\frac{add+cdd-acc-c^3}{4ad-2cc}$. Now, if neither the Terms $aa-cc$ and $2a-2c$ had not had a common Divisor, the propos'd Fraction would have been irreducible.

And this is a general Method of finding common Divisors; but most commonly they are more expeditiously found by seeking all the prime Divisors of either of the Quantities; that is, such as cannot be divided by others, and then by trying if any of them will divide the other without a Remainder. Thus, to reduce $\frac{a^3-aab+abb-b^3}{aa-ab}$ to the least Terms, you must find the Divisors of the Quantity $aa-ab$, viz. a and $a-b$; then you must try whether either a, or $a-b$, will also divide $a^3-aab+abb-b^3$ without any Remainder.

The End of the Third PART.

ARITHMETICAL INSTITUTIONS.

PART IV.

Of the Doctrine of EQUATIONS.

CHAP. I.

Of Equations *in General.*

DEFINITION I.

425. AN *Equation* is the Expreſſion of the Equality between two or more Quantities, whereof one or more is unknown.

DEFINITION II.

426. The *Regiſtring* an *Equation* is the noting down in the *Margin* how it is formed from one or more preceding ones; as in the following Proceſs: where note, that if any Number be inſerted in the Regiſter which is not the Number of ſome foregoing Step, it is diſtinguiſhed by a Line drawn over its Head, as the Number 5 in the 9th Step following.

Regiſter.	Steps.	Proceſs.	Explanation.
	1	$a = 24$	The firſt Step.
	2	$e = 8$	The ſecond Step.
$1 + 2$	3	$a+e=32$	1ſt Step added to the Second.
$1 - 2$	4	$a-e=16$	2d Step ſubtracted from the Firſt.
1×2	5	$ae= 192$	1ſt Step multiplied into the Second.
$1 \div 2$	6	$\frac{a}{e}= 3$	1ſt Step divided by the Second.
2 ☉ ^{2}	7	$ee= 64$	2d Step ſquared.
4 ɯ ^{2}	8	$\overline{a-e}^{\frac{1}{2}}=4$	The Square Root of the 4th Step.
$2 - \overline{5}$	9	$e-5=3$	2d Step leſs $\overline{5}$
6, 9,	10	$\frac{a}{e}=e-5$	6th Step compared with the 9th.

A And

And this Method of Regiſtring Operations was firſt introduced by the ingenious Dr. *John Pell.*

Definition III.

427. *Reduction* of *Equations* is the bringing an unknown Quantity to one ſide, that its Value may be diſcovered; and is performed ſix ſeveral Ways, *viz.* by *Addition*, *Subtraction*, *Multiplication*, *Diviſion*, *Involution*, and *Evolution.*

Definition IV.

428. *Reduction* by *Addition* is the tranſpoſing or removing a Defective Quantity to the contrary ſide of the given Equation, with the Sign + before it. *Ex. gr.*

If	1	$a-x=z$
then becauſe	2	$x=x$
∴	3	$a=z+x$ (In. 60, 68.)

Definition V.

429. *Reduction* by *Subtraction*, is the removing a Poſitive Quantity to the contrary ſide of the Equation, with the Sign — before it. *Ex. gr.*

If	1	$a+x=z$
then becauſe	2	$x=x$
∴	3	$a=z-x$ (In 63, 68.)

Corollary I.

430. Hence it is plain that any Quantity may be tranſpoſed to the contrary ſide by only changing its Sign.

Corollary II.

431. And if the ſame Quantity be affected with the ſame Sign on both ſides an Equation, or with different Signs on the ſame ſide, it deſtroys it ſelf. *Ex. gr.* If $b+2c=b+y$, then by ſubtracting b from both ſides of the Equation, $2c=y$: So if $y-a=r-a$ by adding a to both ſides, $y=r$.

Corollary III.

432. If all the Terms in an Equation be tranſpoſed to the ſame ſide, they will equal 0. *Ex. gr.* If $a=x-z$, then $a-x+z=0$. If $aa=2ba+bb$ then $aa-2ba-bb=0$ (In. 64.)

Definition

Definition VI.

433. *Reduction by Multiplication* is the bringing an Equation out of Fractions, by multiplying their Denominators into every Term of the Equation. *Ex. gr.*

$$
\begin{array}{r|c|l}
 & 1 & \frac{a}{b} = \frac{dd}{bc} - \frac{ad}{bb} \\
1 \times b & 2 & a = \frac{dd}{c} - \frac{ad}{b} \quad \text{(In. 71.)} \\
2 \times c & 3 & ca = dd - \frac{acd}{b} \quad \text{(In. 71.)} \\
3 \times b & 4 & bca = bdd - acd \\
4 + acd & 5 & bca + acd = bdd. \quad \text{(In. 428.)}
\end{array}
$$

Definition VII.

434. *Reduction by Division* is performed by dividing the highest Power of the unknown Quantity by every Factor into which it is multiplied, and every Term in the Equation by every Factor which can be found in them all.

Example 1.

$$
\begin{array}{r|c|l}
 & 1 & b^2a^2 - bca = bd - b \\
1 \div b & 2 & ba^2 - ca = d - 1 \quad \text{(In. 77.)} \\
2 \div b & 3 & a^2 - \frac{c}{b}a = \frac{d-1}{b} \quad \text{(In. 77.)}
\end{array}
$$

Example 2.

$$
\begin{array}{r|c|l}
 & 1 & b^2a^2 - bca^2 = bd - b \\
1 \div b & 2 & ba^2 - ca^2 = d - 1 \\
2 \div \overline{b-c} & 3 & a^2 = \frac{d-1}{b-c}
\end{array}
$$

Definition VIII.

435. *Reduction* by *Involution* is the bringing an Equation out of Surds by transposing the Surd Quantity to one side, and then involving each side as the fractional Exponent shall direct.

Example

Example 1.

Suppose	1	$\overline{a^2x+2axx-x^3}^{\frac{1}{3}}+x=a$
$1-x$	2	$\overline{a^2x+2axx-x^3}^{\frac{1}{3}}=a-x$
$2\circledcirc^3$	3	$a^2x+2axx-x^3=a^3-3a^2x+3ax^2-x^3$ (In. 146.)

Example 2.

	1	$y=\overline{ay+yy-a\overline{ya-yy}^{\frac{1}{2}}}^{\frac{1}{2}}$
$1\circledcirc^2$	2	$yy=ay+yy-a\overline{ya-yy}^{\frac{1}{2}}$ (In. 146.)
$2-yy$	3	$0=ay-a\overline{ya-yy}^{\frac{1}{2}}$
$3+a\overline{ya-yy}^{\frac{1}{2}}$	4	$a\overline{ya-yy}^{\frac{1}{2}}=ay$, or $\overline{ya-y^2}^{\frac{1}{2}}=y$ (In. 434.)
$4\circledcirc^2$	5	$ya-y^2=y^2$ (In. 146.)
$5+y^2$	6	$ya=2y^2$
$6\div y$	7	$a=2y$

DEFINITION. IX.

436. *Reduction by Evolution* is performed by bringing all the Powers of the Quantity sought to one side of the Equation, and then extracting the Root, as the highest Exponent of the unknown Quantity shall direct.

Example 1.

Suppose	1	$aa+cc=bb-2ac$
$1+2ac$	2	$aa+2ac+cc=bb$
$2\omega^2$	3	$a+c=b$ (In. 146.)
$3-c$	4	$a=b-c$

Examples 2.

	1	$a^3=\dfrac{x^3+3x^2z+3xz^2+z^3}{x}$
$1\omega^3$	2	$a=\dfrac{x+z,}{x^{\frac{1}{3}}}$ (In. 146 & 158.)

Example

Example 3.

Suppose	1	$\overline{cc-\frac{3ccaa+2ccca^2}{bb}}^{\frac{1}{2}}+\frac{ac}{b}=\frac{cc}{b}$
Or	2	$\frac{c}{b}\overline{bb-3a^2+2ca^2}^{\frac{1}{2}}+\frac{ac}{b}=\frac{cc}{b}$ (In. 418.)
$2\div\frac{c}{b}$	3	$\overline{bb-3a^2+2ca^2}^{\frac{1}{2}}+a=c$
$3-a$	4	$\overline{bb-3a^2+2ca^2}^{\frac{1}{2}}=c-a$
$4 \odot^2$	5	$bb-3a^2+2ca=c^2-2ca+a^2$
$5+3a^2$	6	$bb+2ca=c^2-2ca+4a^2$
$6-2ca$	7	$bb=cc-4ca+4a^2$
$7\,\mathrm{ɯ}^2$	8	$b=c-2a$
$8+2a-b$	9	$2a=c-b$
	10	$a=\frac{c-b}{2}$

PARTITION I.

438. The prime Diſtinction of *Equations* is into *Final* and *Mixed*.

CHAP. II.

Of Final Equations in General.

DEFINITION X.

437. A *Final Equation* is that which contains in it but one unknown Quantity, of which kind are all the Examples hitherto brought.

DEFINITION XI.

439. In every *Equation* where ever a known Quantity is multiplied into an unknown one, the former is called the *Coefficient* of the latter: as the Quantities n, p, in the Equations $a^3+na^2+pa=q$.

Definition XII.

440. *Final Equations* are said to be of *One*, *Two*, *Three*, &c. *Dimensions*, according as the highest Powers of their unknown Quantities are of *One*, *Two*, *Three*, &c. *Dimensions*: Thus $a+b=c$ is an Equation of *One Dimension*; $aa+a=b$ an Equation of *Two Dimensions*; $a^3+a^2=b$ of *Three*; $a^4=b-c$ of *Four*, &c.

Partition II.

441. And those Equations which are only of one *Dimension* are termed *Simple*, the rest *Compound*.

Definition XIII.

442. That *Compound Equation* is said to be *Quadratic*, where the highest Dimension of the unknown Quantity is a Square; that *Cubic*, where it is a Cube; that *Biquadratic*, where it is a Biquadrate; &c.

Definition XIV.

443. The Root or first Power of the unknown Quantity is called the *Root* of the *Equation*.

Definition XV.

444. And the Sum of all the known Quantities, which are not multiplied into the Root or any Power of it under their proper Signs, is called the *Absolute Number*.

Partition III.

445. *Compound Equations* are either *Inadfected* or *Adfected*.

Definition XVI.

446. A *Compound Inadfected Equation* is that which has its unknown Quantity of the same Dimension in every Term where it occurs; as in these $a^2-b=c$, $xa^3-za^3=d$ or $\begin{smallmatrix}+x\\-z\end{smallmatrix}a^3=d$, &c.

Corollary IV.

447. Therefore the Management of *Compound Inadfected Equations* is directly as *Simple Equations*, only extracting the Root besides, at the End of the Reduction, according as the Exponent shall direct.

Definition

Definition XVII.

448. An *Adfected Equation* is that which has its unknown Quantity in different Dimenſions.

Partition IV.

449. *Adfected Equations* are either *Compleat* or *Incompleat*.

Definition XVIII.

450. A *Compleat Adfected Equation* has all the inferior Powers to the higheſt of the unknown Quantity compleat, as in theſe $2aa=16-14a$, $4a^3+a=22+3aa$, $a^4-6a^2=78-2a^3+a$, $2a^5-3a^4+6a^3-a^2+a=62$, &c.

Definition XIX.

451. In all *Adfected Equations* let the Terms be diſpoſed to the ſame ſide of the Equation, according to the Dimenſions of the unknown Quantity, *viz.* let the higheſt Dimenſions be placed the firſt to the Left-hand, the next higheſt the ſecond, and ſo on, in ſnch ſort that the *Abſolute Number* be always the laſt Term, and if the firſt Term have a Coefficient, divide all by that. And this I call *Preparing an Equation*; *Ex. gr.* the Equation $4a^3+a=22+3a^2$ by due *Preparation* becomes $a^3-\frac{3}{4}a^2+\frac{1}{4}a=\frac{22}{4}=\frac{11}{2}$, or $a^3-\frac{3}{4}a^2+\frac{1}{4}a,-\frac{11}{2}=0$ (In. 432.) So $2a^5-3a^4+6a^3-a^2+a=62$ becomes $a^5-\frac{3}{2}a^4+3a^3-\frac{1}{2}a^2+\frac{1}{2}a-31=0$.

Scholium I.

452. And note (in this Preparation) if the higheſt Dimenſion of the unknown Quantity be defective, all the Terms are to be tranſpoſed to the contrary ſide; *Ex. gr.* the Equation, $16-2aa=14a$, becomes $a^2+7a-8=0$. Or if you pleaſe the *Abſolute Number* may be ſuppoſed to be multiplied into $a^0=1$ (In. 154.) thus $a^2+7a^1-8a^0=0$.

Corollary V.

453. Whence every *Compleat Adfected Equation* is of one Term more than the Number of its Dimenſions (In. 407.)

Definition XX.

454. An *Incompleat Adfected Equation*, is that which wants ſome of the inferior Powers of its unknown Quantity to render it *Compleat*; as the Equations $a^3-a=10$ or $a^3-a-10a^0=0$, where the ſecond Term a^2 is wanting: And

And $3a^4=1925-2a^2$ or by Preparation $a^4+\frac{2}{3}a^2-641\frac{2}{3}=0$, where the second Term a^3 and the fourth Term a^1 are wanting.

Corollary VI.

455. Whence an *Incompleat Quadratic Equation* cannot be *Adfected.*

Scholium II.

456. Every *Compound Equation* may be considered as a *Compleat Adfected* one, by supplying the Term or Terms which are wanting with a Cypher before them, thus, $a^2-bb=0$ is $a^2+0a^1-bba^0=0$; $a^3-a^1-10a^0=0$ is $a^3+0a^2-a^1-10a^0=0$; $a^4+\frac{2}{3}a^2-\frac{1925}{3}=0$ is $a^4+0a^3+\frac{2}{3}a^2+0a^1-\frac{1925}{3}a^0=0$; $a^3-b^3=0$ is $a^3+0a^2+0a^1-b^3=0$, &c.

Corollary VII.

457. If therefore a be put for the Root, m for the Exponent of its highest Power; n for the Sum of all the Coefficients of a^{m-1} under their proper Signs; p for the Sum of all the Coefficients of a^{m-2}; q of a^{m-3}; r of a^{m-2}, &c. then all Compound Equations whatever, after due Preparation, will be universally represented by $a^m \pm na^{m-1} \pm pa^{m-2} \pm qa^{m-3} \pm ra^{m-4} \pm sa^{m-5}$, &c. $=0$ *i. e.* all Quadratic Equations by $a^2 \pm na^1 \pm pa^0=0$; all Cubics by $a^3 \pm na^2 \pm pa^1 \pm qa^0=0$: all Biquadratics by $a^4 \pm na^3 \pm pa^2 \pm qa^1 \pm ra^0=0$, &c. according as m is put for 1, 2, 3, 4, &c.

Partition V.

458. *Prepared Equations* in regard to their Roots, may be distinguished into *Explicable*, and *Inexplicable*.

CHAP. III.

Of Explicable Equations.

Definition XXI.

459. AN *Explicable Equation*, is that which consists of as many Roots as Dimensions, *Ex. gr.* Suppose the unknown Quantity a to have two values, *viz.* $a=\pm b$, and $a=\pm c$; whence $a\mp b=0$ and $a\mp c=0$, then the Product

Product of $\overline{a\mp b=0}\times\overline{a\mp c=0}$ will compose an universal Quadratic Equation, $a^2\mp ba\mp ca\mp bc=0$ or $a^2\mp\begin{matrix}b\\c\end{matrix}a\mp bc=0$ (In. 457.) whose Roots are $\pm b$ and $\pm c$, and $\mp b\mp c=\mp n$ and $\mp bc=\mp p$.

Suppose a again to have a third Value, *viz.* $a=\pm d$, whence $a\mp d=0$, then this multiplied into the Quadratic Equation will compose an universal Cubic Equation, $a^3\mp\begin{matrix}b\\c\\d\end{matrix}a^2\mp\begin{matrix}bc\\bd\\cd\end{matrix}a\mp bcd=0$ (In. 457.) whose Roots are $\pm b$, $\pm c$ and $\pm d$, and $\mp b\mp c\mp d=\mp n$, $\mp bc\mp bd\mp cd=\mp p$, and $\mp bcd=\mp q$.

Again, suppose a to have a fourth Value, *viz.* $a=\pm f$, whence $a\mp f=0$, then this multiplied into the Cubic Equation will compose an universal Biquadratic Equation $a^4\mp\begin{matrix}b\\c\\d\\f\end{matrix}a^3\mp\begin{matrix}bc\\bd\\bf\\cd\\cf\end{matrix}a^2\mp\begin{matrix}bcd\\bcf\\bdf\\cdf\end{matrix}a\mp bcdf=0$ (In. 457.) whose Roots are $\pm b$, $\pm c$, $\pm d$, and $\pm f$; and $\mp b\mp c\mp d\mp f=\mp n$, $\mp bc\mp bd\mp bf\mp cd\mp cf=\mp p$, $\mp bcd\mp bcf\mp bdf\mp cdf=\mp q$, and $\mp bcdf=\mp r$.

And thus by supposing a to have a fifth, sixth, seventh, &c. Value may universal Equations be raised of five, six, seven, &c. Dimensions. And all Equations thus composed are termed *Explicable*.

Corollary VIII.

460. Hence in every *Explicable Equation* the Co-efficient n of the second Term is equal to the Sum of the Roots affected with the contrary Signs; the Co-efficient of the third Term p is equal to the Sum of the Products of every two Roots; the Co-efficient of the fourth Term q to the Sum of the Products of every three Roots; the Co-efficient of the fifth Term r to the Sum of the Products of every four Roots, &c. to the last Term or absolute Number, which is equal to the Product of all the Roots multiplied one into another.

For the Illustration of which take the following Examples:

Example 1.

$a=+b$	1	$a-b=0$
$a=-c$	2	$a+c=0$
$1\times+a$	3	a^2-ba
$1\times+c$	4	$+ca-bc$
$3+4$	5	$a^2\begin{array}{l}-b\\+c\end{array}a-bc=0$ (In. 71.)
$a=+d$	6	$a-d=0$
$5\times+a$	7	$a^3\begin{array}{l}-b\\+c\end{array}a^2-bca$
$5\times-d$	8	$-da^2\begin{array}{l}-bd\\-dc\end{array}a+bcd$
$7+8$	9	$a^3\begin{array}{l}-b\\+c\\-d\end{array}a^2\begin{array}{l}-bc\\+bd\\-cd\end{array}a+bcd=0$
$a=-f$	10	$a+f=0$
$9\times a$	11	$a^4\begin{array}{l}-b\\+c\\-d\end{array}a^3\begin{array}{l}-bc\\+bd\\-cd\end{array}a^2+bcda$
$9\times f$	12	$+fa^3\begin{array}{l}-bf\\+cf\\-df\end{array}a^2\begin{array}{l}-bcf\\+bdf\\-cdf\end{array}a+bcdf$
$11+12$	13	$a^4\begin{array}{l}-b\\+c\\-d\\+f\end{array}a^3\begin{array}{l}-bc\\+bd\\-bf\\-cd\\+cf\\-df\end{array}a^2\begin{array}{l}+bcd\\-bcf\\+bdf\\-cdf\end{array}a+bcdf=0$

Example 2.

$a=b$	1	$a-b=0$
$a=-b$	2	$a+b=0$
$1\times a$	3	a^2-ba
$1\times b$	4	$+ba-bb$
$3+4$	5	$\left\{\begin{array}{l}a^2+0ba-bb=0\\=a^2-b^2\end{array}\right.$
$a=b$	6	$a-b=0$
$5\times a$	7	a^3-b^2a
$5\times-b$	8	$-ba^2+b^3$
$7+8$	9	$a^3-ba^2-b^2a+b^3=0$

If

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1	$b+c+d+f$ &c
2	$bc+bd+bf+cd$
3	$bcd+bcf+bdf$–
4	$bcdf+bcdg+bc$
	&c. &c.
5	$b^1+c^1+d^1+$
6	$b^2+c^2+d^2+$
7	$b^3+c^3+d^3+$
8	$b^4+c^4+d^4+$
9	$b^5+c^5+d^5+f$
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Example 1.

$a=+b$	1	$a-b=0$
$a=-c$	2	$a+c=0$
$1\times+a$	3	a^2-ba
$1\times+c$	4	$+ca-bc$
$3+4$	5	$a^2\begin{smallmatrix}-b\\+c\end{smallmatrix}a-bc=0$ (In. 71.)
$a=+d$	6	$a-d=0$
$5\times+a$	7	$a^3\begin{smallmatrix}-b\\+c\end{smallmatrix}a^2-bca$
$5\times-d$	8	$-da^2\begin{smallmatrix}-bd\\-dc\end{smallmatrix}a+bcd$
$7+8$	9	$a^3\begin{smallmatrix}-b\\+c\\-d\end{smallmatrix}a^2\begin{smallmatrix}-bc\\+bd\\-cd\end{smallmatrix}a+bcd=0$
$a=-f$	10	$a+f=0$
$9\times a$	11	$a^4\begin{smallmatrix}-b\\+c\\-d\end{smallmatrix}a^3\begin{smallmatrix}-bc\\+bd\\-cd\end{smallmatrix}a^2+bcda$
$9\times f$	12	$+fa^3\begin{smallmatrix}-bf\\+cf\\-df\end{smallmatrix}a^2\begin{smallmatrix}-bcf\\+bdf\\-cdf\end{smallmatrix}a+bcdf$
$11+12$	13	$a^4\begin{smallmatrix}-b\\+c\\-d\\+f\end{smallmatrix}a^3\begin{smallmatrix}-bc\\+bd\\-bf\\-cd\\+cf\\-df\end{smallmatrix}a^2\begin{smallmatrix}+bcd\\-bcf\\+bdf\\-cdf\end{smallmatrix}a+bcdf=0$

Example 2.

$a=b$	1	$a-b=0$
$a=-b$	2	$a+b=0$
$1\times a$	3	a^2-ba
$1\times b$	4	$+ba-bb$
$3+4$	5	$\left\{\begin{matrix}a^2+0ba-bb=0\\=a^2-b^2\end{matrix}\right.$
$a=b$	6	$a-b=0$
$5\times a$	7	a^3-b^2a
$5\times-b$	8	$-ba^2+b^3$
$7+8$	9	$a^3-ba^2-b^2a+b^3=0$

If

D

1	$b+c+d+f$ &c
2	$bc+bd+bf+cd$
3	$bcd+bcf+bdf-$
4	$bcdf+bcdg+bc$
&c.	&c. &c.
5	$b^1+c^1+d^1+$
6	$b^2+c^2+d^2+$
7	$b^3+c^3+d^3+$
8	$b^4+c^4+d^4+$
9	$b^5+c^5+d^5+$
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$a = +$
$a = -$
$1 \times + a$
$1 \times + c$

$3 + 4$

$a = +$

$5 \times +$

$5 \times - d$

$7 + $

$a = -$

$9 \times a$

$9 \times f$

$11 + 1$

$a = b$
$a = -b$
$1 \times a$
$1 \times b$

$3 + 4$

$a = b$

$5 \times a$	7	$a^3 - b^2 a$
$5 \times -b$	8	$-ba^2 + b^3$
$7 + 8$	9	$a^3 - ba^2 - b^2 a + b^3 = 0$

If

If in the firſt Example above $+b=+2$, $-c=-3$, $+d=+4$, $-f=-5$, then the Quadratic Equation (Step 5.) will be $a^2+a-6=0$: the Cubic Equation (Step 9.) will be $a^3-3a^2-10a+24=0$: the Biquadratic Equation (Step 13.) will be $a^4+2a^3-25a^2-26a+120=0$. And each of theſe Roots or Values of a ſubſtituted in the laſt Equation will make the whole equal to nothing.

Corollary IX.

461. Hence whenever one Root of any Equation is found, if we divide by that Root, the Equation will ſtill be reduced to one Dimenſion lower. *Ex. gr.* in the Equation $a^3+2a^2-5a-6=0$, if we have one Root found, *viz.* $a=-1$, or $a+1=0$, then dividing by $a+1=0$, the Quotient will be $a^2+a-6=0$ the Cubic Equation reduced to a Quadratic. And if again this latter Equation be divided by $a=2$ or $a-2=0$, the Reſult will be $a+3=0$, or $a=-3$ the Quadratic Equation reduced to a Simple one.

Problem I.

462. In any *Explicable Equation* $a^m \pm na^{m-1} \pm pa^{m-2} \pm qa^{m-3} \pm ra^{m-4}$ &c. $=0$ from the given Coefficients n, p, q, r, s, &c. to find the Sum of the Squares, Cubes, Biquadrates, &c. of their Roots, under their proper Signs.

Effection.

For n put $A=$ the Sum of all the Roots under its contrary Sign (In. 461.) and make $B=$ the Sum of the Squares of the ſaid Roots, $C=$ the Sum of all the Cubes, $D=$ the Sum of all the Biquadrates, $E=$ the Sum of all the Fifth Powers, &c. then I ſay that $nn-2p=B$, or $nA-2p=B$, $nB-pA+3q=C$, $nC-pB+qA-4r=D$, $nD-pC+qB-rA+5s=E$, &c.

You have the Demonſtration of this in the annexed Scheme.

Scholium III.

463. But note, that becauſe A the Sum of all the 1ſt Powers is here taken under the contrary Sign (In. 461.) therefore the Sums of all the odd Powers, *viz.* C, E, G, I, *&c.* (or the 3d, 5th, 7th, 9th, *&c.* Powers) are by the foregoing Problem always exhibited under their contrary Signs.

Ex. gr. Suppoſe the Equation $a^4-2a^3-13a^2+14a+24=0$, whoſe Roots are $-1+2-3+4=A=+2$, conſequently $B=+1+4+9+16=+30$, $C=-1+8-27+64=+44$, $D=+1+16+81+256=+354$. $E=1+32-243+1024=+812$, $F=+1+64+729+4096=+4890$, *&c.* But by the Coefficients $A=n=-2$, $p=-13$, $q=+14$, $r=+24$, whence $B=nA-2p=\overline{|-2\times-2|}-\overline{|-26}=+4+26=+30$ (In. 383, 392.) $C=nB-pA+3q=\overline{|-2\times+30|}-\overline{|-13\times+2|}+42=-60-26+42=-44$: $D=nC-pB+qA-4r=\overline{|-2\times-44|}-\overline{|-13\times+30|}-\overline{|-14\times-2|}-96=+88+390-28-96=+354$: $E=nD-pC+qB-rA+5s=\overline{|-3\times+354}-\overline{|-13\times-44}+14\times30-\overline{|+24\times-2}+5\times0=-708-572+420+48=-812$: $F=nE-pD+qC-rB+sA-6t=\overline{|-2\times-812|}-\overline{|-13\times+354|}+\overline{|+14\times-44|}-24\times30=1624+4602-616-720=+4890$: *&c.*

Again ſuppoſe the Equation $a^3+2a-5a-6=0$ whoſe Roots are $-1+2$ and -3, whence $A=-2$, $B=+14$, $C=-20$, $D=+98$, *&c.* But by the Coefficients, $A=n=+2$, $p=-5$, $q=-6$, whence $B=+14$, $C=+20$, $D=+98$, *&c.*

Laſtly ſuppoſe the Equation $a^3-19a+30=0$, or $a^3+0a^2-19a+30=0$ whoſe Roots are $+2+3-5$, whence $A=0$, $B=30$, $C=-90$, $D=722$, *&c.* But by the Coefficients $A=n=0$, $B=38$, $C=90$, $D=722$, *&c.*

Corollary X.

464. Hence we may learn a Method to find the Roots of *Explicable Equations* nearly, as follows. All 2d, 4th, 6th, 8th, *&c.* Powers are Poſitive Quantities, whether their Roots be Poſitive or Defective (In. 393.) conſequently the Terms B, D, F, H, *&c.* found as above, are every one greater than the reſpective Homologous Powers raiſed from the greateſt Root of the given Equation, whether that Root be Poſitive or Defective : Or, which is the ſame thing $B^{\frac{1}{2}}$ $D^{\frac{1}{4}}$ $F^{\frac{1}{6}}$ $H^{\frac{1}{8}}$ *&c.* are every one greater than the ſaid greateſt Root

Root (In. 22.) But $D^{\frac{1}{4}}$ is nearer equal to it than $B^{\frac{1}{2}}$, $F^{\frac{1}{6}}$ than $D^{\frac{1}{4}}$, $H^{\frac{1}{8}}$ than $F^{\frac{1}{6}}$, $K^{\frac{1}{10}}$ than $H^{\frac{1}{8}}$, *&c. ad infinitum:* Whence it is eaſy to conceive, how the greateſt Root of any given Equation may be approached to, nearer and nearer. *Ex. gr.* Suppoſe the Equation $a^4-2a^3-13a^2+14a+94=0$ were given to be reſolved. Here $B=30$, $D=354$, $F=4890$, $H=72354$, *&c.* therefore $B^{\frac{1}{2}}=5.\ 4$ *&c.* $D^{\frac{1}{4}}=4.33$ *&c.* $F^{\frac{1}{6}}=4.09$ *&c.* $H^{\frac{1}{8}}=4.04$ *&c.* and conſequetny if the Root be an Integer it cannot exceed ± 4. I try therefore by ſubſtituting $+4$ for the Root which does not ſucceed, but -4 ſucccceeds. And the Equation being divided by $a+4=0$ is reduced to this Cubic one $a^3+2a^2-5a-6=0$.

Problem II.

465. In a given *Explicable Equation*, to find how many of its Roots are Poſitive, and how many Defective.

Effection.

When the Equation is prepared, as is directed above, begin at the Left-hand, and count how Changes there are in the Signs from $+$ to $-$ and from $-$ to $+$, and as many Changes as there are, ſo many are the Poſitive Roots; and as many Succeſſions as there are of the ſame Sign without Change, ſo many are the Defective Roots. *Ex. gr.* In the Equation $a^4+2a^3-25a^2-26a+120=0$ the Signs are $++--+$, which ſhews that there are two Poſitive Roots, becauſe there are two Changes of the Signs, *viz.* from $+$ to $-$, and from $-$ again into $+$; alſo two Defective ones, becauſe there are two Succeſſions of the ſame Sign; *viz.* $++$ and $--$: Again, in the Equation $a^6-29a^4+244a^2-576=0$, or $a^6+0a^5-29a^4-0a^3+244a^2+0a-576=0$, the Signs are $++--++-$, which ſhews that there are three Poſitive Roots and three Defective ones, becauſe there are three Changes of the Signs, and three Succeſſions of the ſame Sign. Where note, that in this Caſe the Signs of the inſignificant Terms, or thoſe which are taken to fill up the Equation, as the Terms $0a^5$, $0a^3$, $0a$ above, muſt always be of the ſame Affection with the ſignificant Term immediately foregoing.

Corollary XI.

466. If therefore the Roots of a given Equation be Rational, they may be diſcovered by ſeeking what two Factors, if the Equation be Quadratic; what three, if it be Cubic; what four, if it be Biquadratic, *&c.* do make a Product equal to the laſt Term, and a Sum equal to the Coefficient of the ſecond Term with its contrary Sign: The Factors thus found are the Roots ſought, and by ſubſtituting each in the Equation, will make the Whole equal to nothing.

Example 1.

Let it be required to find the Roots of the Equation $a^2+6a-16=0$. Here the Number of the Roots are two (In. 459.) the one Positive and the other Defective (In. 465.) then all the Divisors in the Absolute Number 16 are 1, 2, 4, 8, 16, among which I am to find two, whose Sum will make -6 the Coefficient of the second Term with its contrary Sign (In. 463.) but there are no other two among them of contrary Affections whose Sum will make -6, and Product -16, but $+2$ and -8, therefore $+2$ and -8 are the Roots required: for $\overline{a-2}=0\times\overline{a+8}=0=a^2+6a-16=0$. And if $+2$ and -8 be each substituted for a, it will be,

In the former Case.

$$\begin{array}{c|l}
1 & a^2=+2^2=4 \\
2 & +6a=+6\times2=+12 \\
3 & -16=-16=-16 \\
\hline
 & a^2+6a-16=0=0
\end{array}$$

In the latter Case.

$$\begin{array}{c|l}
1 & a^2=-8\times-8=+64 \\
2 & +6a=+6\times-8=-48 \\
3 & -16=-16=-16 \\
\hline
 & a^2+6a-16=0
\end{array}$$

Example 2.

Let it be required to find the Roots of the Equation $a^3-a^2-17a-15=0$. Here the Number of Roots are three (In. 459.) one Positive and two Defective (In. 465.) then because $15=1\times3\times5$, and $-1-3+5=+1$ the Coefficient of the second Term with its contrary Sign; therefore -1, -3, and $+5$ are the Roots required for $\overline{a+1}=0\times\overline{a+3}=0\times\overline{a-5}=0=a^3-a^2-17a-15=0$. And if -1, -3, $+5$ be each substituted for a, it will be

$$\begin{array}{l}
+a^3=-1^3=-1 \\
-a^2=-\overline{-1}\times\overline{-1}=-1 \\
-17a=-17\times\overline{-1}=+17 \\
-15=-15=-15 \\
\hline
a^3-a^2-17a-15=0
\end{array}$$

$$\begin{array}{l}
+a^3=-3^3=-27 \\
-a^2=-\overline{-3}\times\overline{-3}=-9 \\
-17a=-17\times\overline{-3}=+51 \\
-15=-15=-15 \\
\hline
a^3-a^2-17a-15=0
\end{array}$$

$$\begin{array}{l}
+a^3=+5^3=+125 \\
-a^2=-\overline{5^2}=-25 \\
-17a=-17\times+5=-85 \\
-15=-15=-15 \\
\hline
a^3-a^2-17a-15=0
\end{array}$$

Scholium IV.

467. The main of what is delivered in this and the two following Chapters, was first invented by that great Improver of Modern Algebra, and Honour to his Country, the celebrated Mr. *Thomas Harriot*; who, as Dr. *Wallis* informs us, died *Anno* 1621, aged about 60 Years, and was interr'd in St. *Christopher's* Church *London*.

CHAP. III.

On Inexplicable Equations.

Definition XXII.

468. Inexplicable *Equations* are such as have not so many Roots as Dimensions,

Partition VI.

469. And these are either *Total* or *Partial*.

Definition XXIII.

470. An Equation *Totally Inexplicable* is that which has no real or possible Roots in Nature, either Positive or Defective, but only such as are supposed to be extracted from a Defective Square, Biquadrate, sixth Power, *&c.* which is impossible (In. 394.) therefore the Roots of such Equations are called impossible or imaginary; whose Composition is as follows. Suppose $a=b-\overline{-z}|^{\frac{1}{2}}$ and $a=b+\overline{-z}|^{\frac{1}{2}}$, or which is the same thing $a+\overline{-z}|^{\frac{1}{2}}-b=0$, and $a-\overline{-z}|^{\frac{1}{2}}-b=0$, then the Product of these two simple Equations will compose the Quadratic Equation $a^2-2ab\begin{smallmatrix}+z\\+bb\end{smallmatrix}=0$

Again, Suppose $a=\overline{-z}|^{\frac{1}{2}}-b$ and $a=-\overline{-z}|^{\frac{1}{2}}$; or $a-\overline{-z}|^{\frac{1}{2}}+b=0$ and $a+\overline{-z}|^{\frac{1}{2}}+b=0$, the Product of these will compose the Quadratic Equation $a^2+2ab\begin{smallmatrix}+z\\+bb\end{smallmatrix}=0$.

471. There-

COROLLARY XII.

471. Therefore all Quadratic Equations of this Form $a \mp 2ab + \overset{z}{bb} = 0$. *i.e.* all which have the abſolute Number Poſitive, and greater than the Square of $\frac{1}{2}$ the Co-efficient of the ſecond Term (or ſuch as have p greater than $\frac{1}{4}\,nn$) are inexplicable.

SCHOLIUM V.

472. Put $BB = bb + z$, then all Inexplicable Quadratic Equations will fall under one of theſe Forms, $a^2 + 2ab + BB = 0$, or $a^2 - 2ab + BB = 0$, and from the Multiplication of theſe together, or the Involution of each ſeparately, all Inexplicable Biquadratic Equations will be repreſented by

$$a^4 \pm 4ba^3 \begin{matrix} +2BB \\ \pm 4bb \end{matrix} a^2 \pm 4bBBa + B^4 = 0$$

All Inexplicable Equations of ſix Dimenſions by

$$a^6 \pm 6ba^5 \begin{matrix} +3BB \\ \pm 12bb \end{matrix} a^4 \pm \begin{matrix} 12bBB \\ 8bbb \end{matrix} a^3 \begin{matrix} +3B^4 \\ \pm 12b^2B \end{matrix} a^2 \pm 6bB^4a + B^6 = 0 \text{ \&c. \&c.}$$

COROLLARY XIII.

473. Hence Equations *totally Inexplicable*, can only be ſuch as are of an even Number of Dimenſions (In. 394.)

DEFINITION XXIV.

474. An Equation Explicable in Part, is that which is compoſed of an Explicable Equation, multiplied into an Inexplicable one; or it is that which has at leaſt one poſſible Root.

COROLLARY XIV.

475. All Equations which conſiſt of an odd Number of Dimenſions have at leaſt one poſſible Root.

PROBLEM III.

476. To know in general how many Roots of any given Equation are impoſſible. *Newton*'s *Algebra*, p. 197.

Effection.

1. If the Equation be incompleat, compleat it (In 456.)
2. Make a Series of Fractions whoſe Denominators are Numbers in this Progreſſion 1, 2, 3, 4, 5, *&c.* going on to the Number which ſhall be the

ſame

same as that of the Dimensions of the Equation; and the Numerators the same Series of Numbers in a contrary Order.

3. Divide each of the latter Fractions by each of the former.
4. Place the Fractions that come out on the middle Terms of the Equation.
5. Under the first and last Terms place the Sign $+$.
6. Under every one of the middle Terms, if its Square multiplied into the Fraction standing over its Head be greater than the Product of the Terms on both Sides, place the Sign $+$; but if it be less, the Sign $-$.
7. Count how many Changes there are in the Series of the underwritten Signs, from $+$ to $-$, and $-$ to $+$; and so many will be the Number of impossible Roots.

Ex. gr. In the Equation $a^3+paa+3ppa-r=0$, I divide the second of the Fractions of this Series $\frac{3}{1}$, $\frac{2}{2}$, $\frac{1}{3}$, *viz.* $\frac{2}{2}$ by the first $\frac{3}{1}$, and the third $\frac{1}{3}$ by the second $\frac{2}{2}$, and I place the Fractions that come out, *viz.* $\frac{1}{3}$ and $\frac{1}{3}$ upon the mean Terms of the Equation, as follows:

$$\begin{array}{cccccc} & \frac{1}{3} & & \frac{1}{3} & & \\ a^3 & +paa & + & 3ppa & -r & =0 \\ + & - & & + & + & \end{array}$$

Then because the Square of the second Term paa multiplied into the Fraction over its Head $\frac{1}{3}$, *viz.* $\frac{ppa^4}{3}$ is less than $3ppa^4$, the Product of the first Term a^3 and third $3ppa$, I place the Sign $-$ under the Term paa. But because $9p^4a^2$ (the Square of the third Term $3ppa$) multiplied into the Fraction over its Head $\frac{1}{3}$ is greater than nothing, and therefore much greater than the negative Product of the second Term paa and the fourth $-r$, I place the Sign $+$ under that third Term. Then under the first Term a^3 and the last $-r$, I place the Sign $+$; and the two Changes of the underwritten Signs, which are in this Series $+-++$, the one from $+$ into $-$, and the other from $-$ into $+$ shew that there are two impossible Roots.

And thus the Equation $a^3-4a^2+4a-6=0$ has two impossible Roots.

$$\begin{array}{cccc} & \frac{1}{3} & \frac{1}{3} & \\ a^3 & -4a^2 & +4a & -6=0 \\ + & + & - & + \end{array}$$

Also the Equation $a^4-6a^2-3a-2=0$ has two

$$\begin{array}{ccccc} & \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & \\ a^4 & +0a^3 & -6a^2 & -3a & -2=0 \\ + & + & + & - & + \end{array}$$

For this Series of Fractions $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, by dividing the second by the first, and the third by the second, and the fourth by the third, gives this Series $\frac{3}{8}$, $\frac{4}{9}$, $\frac{3}{8}$ to be placed upon the middle Terms of the Equation. Then the

 Square

Square of the second Term which is here nothing, multiplied into the Fraction over head, *viz.* $\frac{1}{8}$ produces nothing, which is yet greater than the Defective Product $-6a^6$ contained under the Terms a^4 and $-6aa$. Wherefore under the Term that is wanting I write $+$; in the rest I go on as in the former Example; and there comes out this Series of the underwritten Signs $+\ +\ +\ -\ +$, where two Changes shew there are two impossible Roots. And after the same Way in the Equation $a^5-4a^4+4a^3-2aa-5a-4=0$ are discovered two impossible Roots, as follows:

$$\begin{array}{cccccc} & \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & \\ a^5 & -4a^4 & +4a^3 & -2a^2 & -5a & -4=0 \\ + & + & - & + & + & + \end{array}$$

Scholium VI.

477. Where two or more Terms are at once wanting, under the first of the deficient Terms you must write the Sign $-$, under the second the Sign $+$, under the third the Sign $-$, and so on, always varying the Signs, except that under the last of the deficient Terms you must always place $+$, where the Terms next on both Sides the deficient Terms have contrary Signs. As in the Equations

$$\begin{array}{cccccc} a^5 & +ba^4 & +0a^3 & +0a^2 & +0a & +b^5=0 \\ + & + & - & + & - & + \end{array}$$

And

$$\begin{array}{cccccc} a^5 & +ba^4 & +0a^3 & +0a^2 & +0a & -b^5=0 \\ + & + & - & + & + & + \end{array}$$

the first whereof has four, and the latter two impossible Roots.

$$\begin{array}{cccccccc} & \frac{3}{7} & \frac{5}{9} & \frac{3}{5} & \frac{3}{5} & \frac{5}{9} & \frac{3}{7} & \\ a^7 & -2a^6 & +3a^5 & -2a^4 & +a^3 & +0a^2 & +0a & -3=0 \\ + & - & + & - & + & - & + & + \end{array}$$

has six impossible Roots. *Newton*'s *Algebra*, page 198, 199.

Corollary XV.

478. Hence also may be known whether the impossible Roots are among the Positive or Defective ones. For the Signs of the Terms over head of the subscribed changing Terms shew, that there are as many impossible Positive Roots, as there are Variations of them; and as many Defective ones, as there are Successions without Variation. Thus, in the Equation

$$\begin{array}{cccccc} a^5 & -4a^4 & +4a^3 & -2a^2 & -5a & -4=0, \\ + & + & - & + & + & + \end{array}$$

because

because by the Signs writ underneath that are changeable, *viz.* $+ - +$, by which it is shewn there two impossible Roots, the Terms over-head $-4a^4 + 4a^3 - 2a^2$ have the Signs $- + -$, which by two Variations shew there are two Positive Roots; therefore there will be two impossible Roots among the Positive ones. Since therefore the Signs of all the Terms of the Equation $+ - + - - -$ by three Variations shew that there are three Positive Roots, and that the other two are Defective, and that among the Positive ones there are two impossible ones; it follows that there are, one true Positive Root, two Defective ones, and two impossible ones. Now if the Equation had been

$$a^5 - 4a^4 - 4a^3 - 2a^2 - 5a - 4 = 0,$$
$$+ \quad + \quad - \quad - \quad + \quad +$$

then the Terms over-head of the subscribed former Terms $+ -$, *viz.* $-4a^4 - 4a^3$, by their Signs that don't change $-$ and $-$ shew that one of the Defective Roots is impossible; and the Terms over the foregoing underwritten varying Terms $- +$, *viz.* $-2aa - 5a$, by their Terms not varying, $-$ and $-$, shew that one of the Defective Roots is impossible. Wherefore, since the Signs of the Equation $+ - - - - -$ by one Variation shew there is one Positive Root, and that the other four are Defective, it follows there is one Positive, two Defective, and two impossible ones. And this is so where there are not more impossible Roots than what are discovered by the Rule preceding. For there may be more, tho' it seldom happens. *Newton's* Algebra. Pag. 199, 200.

CHAP. V.

Of the Transmutation of Equations.

DEFINITION XXV.

479. BY Transmutation of Equations is meant the changing them into other Forms, by means whereof their Resolution is oftentimes facilitated.

PROBLEM IV.

480. To change the Positive Roots of an Equation into Defective ones, and the Defective Roots into Positive ones.

Effection.

Effection.

Change the Sign of every other Term, *i. e.* of the second, fourth, and sixth, *&c.* Terms, and it is done. *Ex. gr.* the Equation $a^5+3a^4-23a^3-51a^2+94a+120=0$, by changing the alternate Signs, becomes $a^5-3a^4-23a^3+51a^2+94a-120=0$; the former Equation consists of two Positive and three Defective Roots, and the latter of three Positive, and two Defective Roots (In. 465.) *viz.* the Roots of the former are $+2$, $+4$, -1, -3, -5, of the latter $+1$, $+3$, $+5$, -2, -4, (In. 466.) In like Manner the Equation $a^4+2a^3-25a^2-26a+120=0$ whose Roots are $+2$, $+4$, -3, -5, becomes $a^4-2a^3-25a^2+26a+120=0$ whose Roots are $+3$, $+5$, -2, -4. Again in the Equation $a^5-4a^4+4a^3-2a^2-5a+4=0$ (In. 478.) the three Positive Roots will be changed into Defective ones, and the two Defective ones into Positive, by changing the given Equation into $a^5+4a^4+4a^3+2a^2-5a+4=0$, where the two impossible Roots, which lay hid in the former Equation among the Positive ones, lie hid in this latter among the Defective ones; so that there remains but one Root truly Defective.

PROBLEM V.

481. To augment or diminish the Roots of a given Equation by a given Quantity.

Effection.

1. For the Quaintity, by which the Roots are to be augmented or diminished, put z.
2. Substitute $y-z$ every where in the given Equation for the Root, if it be augmented; or $y+z$ if it be to be diminished, and it is done.

Example 1.

Let it be required to augment the Root a of the Equation $a^3-3a^2-10a+24=0$ by $z=2$.

Make $a+2=y$ or $a=y-2$, whence $a^2=y^2-4y+4$, $a^3=y^3-6y^2+12y-8$, then

	1	$+1a^3=+y^3-6y^2+12y-8=\overline{y-2}^3$
	2	$-3a^2=-3y^2+12y-12=-3\times\overline{y-2}^2$
	3	$-10a=-10y+20=-10\times\overline{y-2}$
	4	$+24=+24$
$1+2+3+4$	5	$a^3-3a^2-10a+24=y^3-9y^2+14y+24=0$ an Equation where $y=a+2$

Example

Example 2.

Let it be required to diminish the Root a of the Equation $a^3-3a^2-10a+24=0$ by $z=3$.

Make $a-3=y$ or $a=y+3$, whence $a^2=y^2+6y+9$, $a^3=y^3+9y^2+27y+27$. Then

$$\begin{array}{r|c|l}
 & 1 & a^3 = \quad y^3+9y^2+27y+27 \\
 & 2 & -3a^2 = \quad -3y^2-18y-27 \\
 & 3 & -10a = \quad -10y-30 \\
 & 4 & +24 = \quad +24 \\
\hline
1+2+3+4 & 5 & a^3-3a^2-10a+24=y^3+6y^2-y-6=0 \text{ an Equation} \\
 & & \text{where } y=a-3.
\end{array}$$

Corollary XVI.

482. While the Positive Roots of an Equation are augmented, the Defective ones are diminished, and *vice versa*, while the Defective Roots are augmented, the Positive ones are diminished. *Ex. gr.* in the Equation $a^3-3a^2-10a+24=0$, whose Roots are $+2$, $+4$, -3, if $y-2$ be substituted for a, then in the resulting Equation $y^3-9y^2+14y+24=0$ the Roots will be $+4$, $+6$, -1; and if $y+3$ be substituted for a in the resulting Equation $y^3+6y-y-6=0$, the Roots will be -1, $+1$, -6.

Corollary XVII.

483. If the Roots of any Equation be augmented by a Quantity greater than its greatest Defective Root, the Defective Roots will become Positive; and on the contrary, if the Roots be diminished by a Quantity greater than the greatest Positive Root, the Positive Roots will become Defective.

Corollary XVIII.

484. From this last Problem we also learn to compleat an Equation, wherein any of the Terms are wanting, by augmenting or diminishing the Root by a given Quantity. *Ex. gr.* suppose the Equation $a^3+5a-20=0$ were to be compleated. Make $a=y-1$, then

$$\begin{array}{ll}
+a^3= & y^3-3y^2+3y-1 \\
+5a= & \qquad\quad +5y-5 \\
-20= & \qquad\qquad\ -20 \\
\hline
\end{array}$$

$a^3+5a-20=y^3-3y^2+8y-26=0$ a compleat Equation wherein $y=a+1$.

Corollary XIX.

485. Hence again we learn to transform any given Equation wherein the absolute Number admits of many Divisors, into another Equation, whose absolute Number has fewer Divisors; which is done by substituting $a=+1$ or -1, $a=+2$ or -2, $a=+3$ or -3, $a=+4$ or -4, &c. successively; and observing which of these substituted Values produces a Number that has fewer Divisors than the absolute Number of the given Equation: And if the Root of the given Equation be augmented or diminished by that Number, the Equation resulting will be one which has an absolute Number of fewer Divisors than the foregoing one. *Ex. gr.* in the Equation $a^3-3a^2-10a+24=0$, make $a=+1$, then will

$$\begin{array}{rl} a^3= & +1 \\ -3a^2= & -3 \\ -10a= & -10 \\ +24= & +24 \\ \hline \text{whose Sum is} & +12 \end{array}$$

Since therefore 12 has fewer Divisors than 24, make $a=y+1$ whence

$$\begin{array}{rr} a^3= & y^3+3y^2+3y+1 \\ -3a^2= & -3y^2-6y-3 \\ -10a= & -10y-10 \\ +24= & +24 \\ \hline \end{array}$$

$a^3-3a^2-10a+24=y^3-13y+12=0$, an Equation wherein $y=a-1$.

Problem VI.

486. To multiply or divide the Root a of any given Equation $a^m \pm na^{m-1} \pm pa^{m-2} \pm qa^{m-3}$ &c. $=0$, by a given Quantity z.

Effection.

1. Multiply or divide the Coefficient of the first Term of the given Equation *viz.* 1 by 1; of the Second, *viz.* n by the given Quantity z; of the third Term, *viz.* p by zz; of the Fourth, *viz.* q by zzz, of the Fifth, *viz.* r by $zzzz$, &c.

2. For a substitute y. Then I say that y, the Root of the new Equation, will be equal to za or $\frac{a}{z}$, according as the Coefficients of the given Equation are multiplied or divided by z. *Ex. gr.* suppose the Root a be to be multiplied by $3=z$ in the given Equation $a^3-a^2-17a-15=0$: Here if $3a=y$, we

we shall have this new Equation $y^3-3\times y^2-9\times 17y-27\times 15=0$, or $y^3-3y^2-153y-405=0$, consequently if the Root a be to divided by $3=z$ in the Equation $a^3-3a^2-153a-405=0$, by putting $\frac{a}{3}=y$ we shall have the Equation $\frac{1}{1}y^3-\frac{1}{3}y^2-\frac{153}{9}y^1-\frac{405}{27}y^0=0$ or $y^3-y^2-17y-15=0$.

Demonstration.

In the Equation $a^m \pm na^{m-1} \pm pa^{m-2} \pm qa^{m-3} \pm ra^{m-4}$ &c. $=0$; if

	1	$az=y$
$1\div z$	2	$a=\frac{y}{z}$
∴	3	$a^m=\frac{y^m}{z^m}$, $a^{m-1}=\frac{y^{m-1}}{z^{m-1}}$, $a^{m-2}=\frac{y^{m-2}}{z^{m-2}}$
And	4	$\frac{y^m}{z^m}\pm n\frac{y^{m-1}}{z^{m-1}}\pm p\frac{y^{m-2}}{z^{m-2}}\pm q\frac{y^{m-3}}{z^{m-3}}\pm r\frac{y^{m-4}}{z^{m-4}}$ &c. $=0$
		Whence
$4\times z^m$	5	$y^m \pm zny^{m-1} \pm z^2py^{m-2} \pm z^3qy^{m-3} \pm z^4ry^{m-4}$ &c. $=0$ an universal Equation wherein $y=az$.

Again if	1	$\frac{a}{z}=y$
$1\times z$	2	$a=yz$
Whence	3	$a^m=y^m z^m$, $a^{m-1}=y^{m-1}z^{m-1}$, $a^{m-2}=y^{m-2}z^{m-2}$, $a^{m-3}=y^{m-3}z^{m-3}$ &c.
And	4	$z^m y^m \pm nz^{m-1}y^{m-1} \pm pz^{m-2}y^{m-2} \pm qz^{m-3}y^{m-3} \pm rz^{m-4}y^{m-4}$ &c. $=0$
		Consequently,
$4\div z^m$	5	$y^m \pm \frac{n}{z}y^{m-1} \pm \frac{p}{zz}y^{m-2} \pm \frac{q}{zzz}y^{m-3} \pm \frac{r}{zzzz}y^{m-4}$ &c. $=0$
		an Equation wherein $y=\frac{a}{z}$

Corollary XX.

487. Hence is learned to free an Equation out of Fractions, by multiplying the Root either by the Product of all the Denominators, or by some Number which measures or is measured by all the Denominators, as follows.

Examples.

$$\begin{array}{r|l} & a^3-\frac{1}{2}a^2+\frac{2}{3}a-4=0 \\ & 1,\quad 6,\quad 36,\quad 216, \\ \hline y=6a & y^3-3y^2+24y-864=0 \end{array}$$

$$\begin{array}{r|l} & a^4+0a^3-\frac{1}{3}a^2+\frac{2}{3}a-1=0 \\ & 1,\quad 3,\quad 9,\quad 27,\ 81, \\ \hline y=3a & y^3+0y^3-3y^2+18y-81=0 \end{array}$$

$$\begin{array}{r|l} & a^2-\frac{1}{4}a-\frac{5}{16}=0 \\ & 1,\quad 4,\quad 16, \\ \hline y=4a & y^2-y-5=0 \end{array}$$

$$\begin{array}{r|l} & 3a^3+0a^2-\frac{4}{5}a-\frac{2}{3}=0 \\ & 1,\ 15,\ 325,\ 4875, \\ \hline y=15a & y^3+0y^2-130y-1350=0 \end{array}$$

Corollary XXI.

488. Also hence we may learn the manner of freeing an Equation out of Surds.

Examples.

$$\begin{array}{r|l} & a^3+a^2\times 2^{\frac{1}{2}}-3a-5\times\overline{2}^{\,\frac{1}{2}}=0 \\ & 1\qquad 2^{\frac{1}{2}}\qquad 2\qquad 2\times\overline{2}^{\,\frac{1}{2}} \\ \hline y=a\times\overline{2}^{\,\frac{1}{2}} & y^3+2y^2-6y-20=0 \end{array}$$

$$\begin{array}{r|l} & a^3-5a^2\times 3^{\frac{1}{3}}+6a\times\overline{3}^{\,\frac{2}{3}}-7=0 \\ & 1\qquad 3^{\frac{1}{3}}\qquad 9^{\frac{1}{3}}\qquad 27^{\frac{1}{3}} \\ & \qquad\qquad \text{or} \\ & 1\qquad 3^{\frac{1}{3}}\qquad 3^{\frac{2}{3}}\qquad 3^{\frac{3}{3}} \\ \hline y=\dfrac{a}{3^{\frac{1}{3}}} & y^3-5y^2+6y\times\overline{3}^{\,\frac{3}{3}}-\frac{7}{3}=0,\ \text{or} \\ & y^3-5y^2+18y-\frac{7}{3}=0 \end{array}$$

$$\left|\begin{array}{l} a^4+0a^3-aa\times 2^{\frac{1}{3}}-a\times 2^{\frac{1}{2}}-5\times 2^{\frac{2}{3}}=0 \\ \text{or} \\ a^4+0a^3-a^2\times 2^{\frac{2}{6}}-a\times 2^{\frac{3}{6}}-5\times 2^{\frac{4}{6}}=0 \\ 1,\quad 2^{\frac{1}{6}},\quad 2^{\frac{2}{6}},\quad 2^{\frac{3}{6}},\quad 2^{\frac{4}{6}} \\ \hline \end{array}\right.$$

$$y=\frac{a}{2^{\frac{1}{6}}}\ \Big|\ y^4+0y^3-y^2-y-5=0$$

Corollary XXII.

489. And hence laſtly is learned to diſtribute a given Equation into Periods, by looking upon the Co-efficient of the ſecond Term as a Lateral, of the third Term as a Square, of the fourth as a Cube, of the fifth as a Biquadrate, &c. and the abſolute Number as a Power of the ſame Denomination with the given Equation *Ex. gr.* the Equation. $a^4-72a^3+1268a^2-14593a+1000000=0$, ~~is thus diſtributed into Periods,~~

$$a^4-72a^3+1268a^2-14593a+1000000=0$$

And if any Co-efficient have not ſo many Periods as the abſolute Number, it may be ſupplied with Cyphers to the left Hand for Integers and the right for Fractions, as in theſe,

$$a^3+01a^2-072a-30753=0$$

$$a^4-3.0aa^2-0.8125=0$$

Problem VII.

490. To take away the ſecond Term na^{m-1} in any given Equation.

Effection.

1. Divide the Co-efficient of the ſecond Term n by the Number of the Dimenſions of the Equation m.

2. Augment or diminiſh the Root a by the Quotient $\frac{n}{m}$ according as the ſecond Term na^{m-1} is Poſitive or Defective, and the Equation reſulting will want its ſecond Term. *Q. E. E.*

Demonſtration.

Suppoſe, *Ex. gr.* the ſecond Term were to be taken away from the Cubic

Equation $a^3-6a^2+17a-38=0$, or $a^3-na^2+17a-38=0$: Make $a=y-e$. Whence

$$\begin{aligned} a^3 &= y^3-3ey^2+3e^2y-e^3 \\ -na^2 &= -ny^2+2ney-ne^2 \\ +17a &= 17y-17e \\ -38 &= -38 \end{aligned}$$

If then in summing up the above-written Values of a, the Terms $-3ey^2$ $-ny^2$, or $-3e-n=0$. Whence $-3e=+n$, or $+3e=-n$. Therefore $e=-\frac{n}{3}=-\frac{6}{3}=-2$; consequently $a=y+2$.

$$\begin{aligned} a^3 &= y^3+6y^2+12y+8 \\ -6a^2 &= -6y^2-24y-24 \\ +17a &= +17y+34 \\ -38 &= -38 \end{aligned}$$

$a^3-6a^2+17a-38 = y^3 \quad +5y-20=0$, an Equation wanting the second Term, wherein $y=a-2$.

And if the second Term of the given Equation had been $+na^2$, e would have been equal to $+\frac{n}{3}$ or $+2$, and consequently for a in such Case might be substituted $y-2$.

Thus also in a Quadratic Equation e will be found $=\pm\frac{n}{2}$, in a Biquadratic $e=\pm\frac{n}{4}$, in an Equation of five Dimensions $e=\pm\frac{n}{5}$, in an Equation of six Dimensions $e=\pm\frac{n}{6}$, *&c. Q. E. D.*

Corollary XXIII.

491. Hence it is plain that if the second Term p be taken away from any Quadratic Equation, it will be reduced to an Inadfected one, and may be resolved as such. *Ex. gr.* suppose $a^2+a-6=0$, a given Quadratic Equation, consisting of a Positive and a Defective Root; make $a=y-\frac{1}{2}$, whence

$+a^2=$

$$\begin{aligned} +a^2 &= yy - y + \tfrac{1}{4} \\ +a &= \phantom{yy-{}} y - \tfrac{1}{2} \\ -6 &= \phantom{yy-y+{}} -6 \end{aligned}$$

$yy - 6\frac{1}{4} = 0$, or $yy = 6.25$; consequently $y = 2.5$ (In. 146.) and $a = y - \frac{1}{2} = 2$ the Positive Value of a. Then $\frac{a^2+a-6}{a-2} = a+3=0$, therefore $a=-3$ the Defective Value of a.

Corollary XXIV.

492. Alfo by taking away the fecond Term, all Adfected Cubic Equations may be reduced to three Forms, *viz.*

$$\left.\begin{aligned} a^3-pa-q&=0 \\ a^3+pa-q&=0 \\ a^3-pa+q&=0 \end{aligned}\right\} \text{ or } \left\{\begin{aligned} a^3-pa&=q \\ a^3+pa&=q \\ a^3-pa&=-q \end{aligned}\right\} \text{ or } \left\{\begin{aligned} a^3&=q+pa \\ a^3&=q-pa \\ a^3&=pa-q \end{aligned}\right.$$

Problem VIII.

493. To take away the third Term qa^{m-2} in a given Equation *Ex. gr.* Suppofe the Equation $a^4-3a^3+3aa-5a-2=0$, and make $a=y-e$, then

$$\begin{aligned} a^4 &= y^4-4ey^3+6e^2y^2-4e^3y+e^4 \\ -3a^3 &= -3y^3+9ey^2-9e^2y+3e^3 \\ +3a^2 &= +3y^2-6ey+3e^2 \\ -5a &= -5y+5e \\ -2 &= -2 \end{aligned}$$

$$a^4-3a^3+3a^2-5a-2=y^4 \begin{matrix}-4e\\-3\end{matrix}\, y^3 \begin{matrix}+6ee\\+9e\\+3\end{matrix}\, y^2 \begin{matrix}-4eee\\-9ee\\-6e\\-5\end{matrix}\, y \begin{matrix}+eeee\\+3eee\\+3ee\\+5e\\-2\end{matrix} =0$$

Becaufe the third Term in this Equation is $+6ee+9e+3$, therefore make $6ee+9e+3=0$, which divided by 6 becomes $ee+\frac{3}{2}e+\frac{1}{2}=0$. Make $e=u-\frac{3}{4}$ then

$$\begin{aligned} ee &= uu-\tfrac{3}{2}u+\tfrac{9}{16} \\ \tfrac{3}{2}e &= \phantom{uu-{}}\tfrac{3}{2}u-\tfrac{18}{16} \\ \tfrac{1}{2} &= \phantom{uu-\tfrac{3}{2}u-{}}\tfrac{8}{16} \end{aligned}$$

And $uu-\frac{1}{16}=0$, or $uu=\frac{1}{16}$, whence $u=\frac{1}{4}$ and confequently $e=\frac{1}{4}-\frac{3}{4}=$

$\frac{1}{4} = -\frac{1}{2}$, and $a = y + \frac{1}{2}$. And if for a in the given Equation be wrote $y + \frac{1}{2}$, there will arise this Equation wanting the third Term $y^4 - y^3 - \frac{11}{4}y - \frac{21}{16} = 0$ *Q. E. F.*

PROBLEM IX.

494. To take away the last Term but one out of an Equation, where the second Term is wanting.

This is performed by only substituting the last Term divided by y for the Root sought.

Ex. gr. Let it be required to take away the last Term but one from the Equation $a^3 + 5a - 20 = 0$. Make $a = \frac{20}{y}$, then $a^3 + 5a - 20 = \frac{8000}{yyy} + \frac{100}{y} - 20 = 0$; which last Equation multiplied by yyy becomes $8000 + 100yy - 20yyy = 0$, and that again divided by 20 becomes $400 + 5yy - yyy = 0$, or $yyy - 5yy - 400 = 0$; an Equation wherein $y = \frac{20}{a}$: *Q. E. E.*

CHAP. VI.

Of the Limits of Equations.

DEFINITION XXVI.

495. THE *Limits* of an Adfected Equation are two Quantities, between which all its Roots are contained: or according to others, they are two Quantities, between which the greatest Positive Root is contained.

PROBLEM X.

496. To find the Limits of an Equation.

Effection from Sir Isaac Newton.

Multiply every Term of the Equation by the Number of its Dimensions, and divide the Product by the Root of the Equation; then again multiply every one of the Terms that comes out, by a Number less by Unity than before, and divide the Product by the Root of the Equation; and so go on, always multiplying by Numbers less by Unity than before, and dividing the Product by the Root, till at length all the Terms are destroyed,

destroyed, whose Signs are different from the Sign of the first or highest Term, except the last, and that Number will be greater than any Affirmative Root; which being writ in the Terms that come out in the Room of the Root, makes the Aggregate (or Sum) of those which were each Time produced by Multiplication to have always the Sign with the first or highest Term of the Equation. As if there was proposed the Equation $a^5-2a^4-10a^3+3a^2+63a-120=0$. I first multiply it thus, $\overset{5}{a^5}-\overset{4}{2a^4}-\overset{3}{10a^3}+\overset{2}{3a^2}+\overset{1}{63a}-\overset{0}{120}=0$. Then I again multiply the Terms that come out divided by a, thus, $\overset{4}{5a^4}-\overset{3}{8a^3}-\overset{2}{30a^2}+\overset{1}{60a}+\overset{0}{63}$. And dividing the Terms which come out again by a, there comes out, $20a^3-24a^2-60a+60$, which, to lessen them, I divide by the greatest common Divisor 4, and you have $5a^3-6a^2-15a+15$. These being again multiplied by the Progression 3, 2, 1, 0, and divided by a, become $5a^2-4a-5$. And these multiplied by the Progression 2, 1, 0, and divided by $2a$ become $5a-2$. Now since the highest Term of the Equation a^5 is Positive, I try what Number wrote in these Products, for a will cause them all to be Positive. And by trying 1, you have $5a-2=+3$ Positive; but $5a^2-4a-5=-4$ Defective. Wherefore the Limit will be greater than 1. I therefore try some greater Number, as 2, and substituting 2 in each for a, they become

$$\begin{array}{lr}
5a-2 & =+08 \\
5a^2-4a-5 & =+07 \\
5a^3-6a^2-15a+15 & =+01 \\
5a^4-8a^3-30a^2+60a+63 & =+79 \\
a^5-2a^4-10a^3+30a^2+63a-120 & =+46
\end{array}$$

Wherefore since the Numbers that come out 8. 7. 1. 79. 46. are all Positive, the Number 2 will be greater than the greatest Positive Root.

In like manner, if I would find the Limit of the Defective Roots, I try Defective Numbers. Or which is all one, I change the Signs of every other Term, and try Positive ones. But having changed the Signs of every other Term, the Quantities in which the Numbers are to be substituted, will become

$$\begin{array}{l}
5a+2 \\
5a^2+4a-5 \\
5a^3+6a^2-15a-15 \\
5a^4+8a^3-30a^2-60a+63 \\
a^5+2a^4-10a^3-30a^2+63a+120
\end{array}$$

Out of these I chuse some Quantity wherein the Defective Terms seem most prevalent; suppose $5a^4+8a^3-30a^2-6a+63$, and here substituting for

a the Numbers 1 and 2, there come out the Negative Numbers —14, and —33. Whence the Limit will be greater than —2. But substituting the Number 3, there comes out the Affirmative Number 234. And in like manner in the other Quantities, by substituting the Number 3 there comes out always an Affirmative Number, which may be seen by bare Inspection. Wherefore the Number —3 is greater than all the Negative Roots. And so you have the Limits +2 and —3; between which are all the Roots. *Newt. Alg. p.* 208, 209.

Scholium VII.

497. The Invention of Limits is of Use both in the Reduction of Equations by Rational Roots, and in the Extraction of Surd Roots out of them; lest we might sometimes go about to look for the Root beyond these Limits. Thus, in the last Equation, if I would find the Rational Roots, if perhaps it has any; from what we have said it is certain they can be no other than the Divisors of the last Term of the Equation which here is 120. Then trying all its Divisors, if none of them wrote in the Equation for *a* will make all the Terms vanish, it is certain, that the Equation will admit of no Root but what is Surd. But there are many Divisors of the last Term 120, *viz.* +1, —1, +2, —2, +3, —3, +4, —4, +5, —5, +6, —6, +8, —8, +10, —10, +12, —12, +15, —15, +20, —20, +24, —24, +30, —30, +40, —40, +60, —60, +120, —120. To try all these Divisors would be too tedious: But it being known that the Roots are between +2 and —3, we are freed from that Labour. For now there will be no need to try the Divisors, unless those only that are within these Limits, *viz.* the Divisors +1, —1, and —2. For if none of these be the Root, it is certain that the Equation has no Root but what is Surd. *Newt. Alg.* ibid. Otherwise the Limis of Equations may be found, as follows.

Problem XI.

498. To determine the Place of the highest Figure of the Positive Root which first results in the Resolution of a given Affected Equation.

Effection.

1. Make the Absolute Number a Defective Quantity, if it be not so already (In 480.)
2. Distribute the given Coefficients, (*viz.* n, p, q, r, &c.) into Periods (In. 489.)
3. See which of these Numbers (n, $p^{\frac{1}{2}}$, $q^{\frac{1}{3}}$, $r^{\frac{1}{4}}$, &c.) consists of the highest Places of Figures; or (which is all one) see which of the Coefficients consists

sists of the highest Period above or below Unity. And if no Coefficient consists of higher Periods than the Absolute Number, we are to conclude that the highest Figure of the Positive Root required (if the Equation admit of a Positive Root) is the same Number of Places above or below Unity with that of the Homologous Root of the Absolute Number. But if any of the Coefficients consist of higher Periods than the highest of the Absolute Number.

4. Substitute for the Root of the Equation some one of the Numbers 1, 10, 100, 1000. *&c.* 0.1, 0.01, 0.001, *&c. viz.* such as shall be one Place higher than the proper Root of the intermediate Coefficient, or Coefficients, of the the highest Periods, which in this Case (if the Equation have a Positive Root) will always make a Positive Result, the first Term being Positive; and a Defective Result the first Term being Defective.

5. Substitute successively every next lower of the above mentioned Numbers 1, 10, 100, 1000, *&c.* 0.1, 0.01, 0.001, *&c.* 'till the Result be equal to or less than nothing, if the first Term be Positive; and equal to or more than nothing, if the first Term be Defective: Or if in the former Case the Result be always Positive, and in the latter always Defective, then mark that Number which makes the Result nearest equal to nothing. And I say, in the former Case, the Number which makes the Result nearest equal to or less than nothing; and in the latter Case, the Number which makes the Result nearest equal to or greater than nothing, will be of the same Place with the highest Figure of the Root required. *Q. E. E.*

Thus in the Equation $a^3-27a^2-1063a-33615=0$; and $-a^4-2a^3+25a^2+26a-120=0$; because the Coefficients of the former are each of two Periods from Unity, and the latter of one, therefore by *Prec.* 3. the first Positive Root of the former Equation will consist of two integral Places; and the first Root of the latter of one; *i. e.* the Root of the former is some Number between 10 and 100, and the Root of the latter between 1 and 10.

Example 3.

In the Equation	1	$a^4-0.03a^2+0.075a-1=0$
subst. by Pre. 4, $a=1$	2	$1-0.03+0.075-1=+0.045>0$
$a=0.1$	3	$0.0001-0.0003+0.0075-1=-0.9927<0$
		Whence a is some Number between 1 and 0.1

Example

Example 4.

In the Equation.	1	$-a^4+80a^3-1998a^2+14937a-5000=0$ the first Term being Defective.
or		$-a^4+80a^3-2000a^2+15000a-5000=0$ near
Sub. Pre. 4, $a=100$	2	$-100000000+80000000-20000000+1500000-5000=-38505000<0$
$a=10$	3	$-10000+80000-200000+150000-5000=+15000>0$

Therefore a is some Number between 100 and 10

Example 5.

In the Equation	1	$a^3+347a^2-69a-0.088=0$, the first Term being Positive.
Substit. $a=1000$	2	$1000000000+347000000-69000-0.088=+446930999.912>0$
$a=100$	3	$1000000+3470000-6900-0.088=+4463099.912>0$
$a=10$	4	$1000+34700-690-0.088=+35009.912>0$
$a=1$	5	$1+347-69-0.088=+278.912>0$
$a=0.1$	6	$0.001+3.47-6.9-0.088=-3.517<0$

Therefore a is between 1 and 0.1.

Example 6.

In the Equation	1	$a^3-18a+24=0$
Substit. $a=10$	2	$1000-180+24=+844>0$
$a=1$	3	$1-18+24=+7>0$
$a=0.1$	4	$0.001-1.8+24=+22.201>0$

Therefore a is some Number between 10 and 0.1 (Pre. 5.)

Example

Example 7.

In the Equation	1	$-a^3+18a-24=0$
Subftit. $a=10$	2	$-1000+180-24=-844<0$
$a=1$	3	$-1+18-24=-7<0$
$a=0.1$	4	$-0.001+1.8-24=-22.201<0$

Therefore a is fome Number between 10 and 1.

SCHOLIUM VIII.

499. The Reafon of thefe Operations will appear plainly to the Learner from a little Practice. And he will readily difcern when feveral of the firft Subftitutions may be omitted, as in the firft Example.

SCHOLIUM IX.

500. If the Equation have no Pofitive Roots, then all the Defective Roots may be changed into Pofitive ones (In. 480.) After which proceed as above.

PROBLEM XII.

501. To find the firft Figure of the firft Pofitive Root in a given Adfected Equation.

Effection.

1. Find the Limits of the Root by the laft Problem.
2. Reject if you pleafe all the Figures of the inferior Periods in every Term beneath the higheft of the Abfolute Number.
3. Subftitute 5 for the Root between the Limits already found, which will difcover whether the Root fought be between that and the leffer Limit, or that and the greater.
4. If the former be the Cafe, fubftitute fucceffively 4, 3, and 2; if the latter, fubftitute 6, 7, 8, 9, and that Number which makes the Refult next leffer than nothing, if the firft Term of the Equation be Pofitive; or next greater, if the firft Term be Defective, will be the firft Figure required.

Example 1.

In the Equation	1	$a^3-28.25a^2-91.75a-62.5=0$ or $a^3-28a^2-92a-60=0$ (Pre. 2.) whofe Limits are 10, 100 (In. 498.)
make $a=50$	2	$125000-70000-4600-60=+50340>0$
$a=40$	3	$64000-44800-3680-60=+15460>0$ near
$a=30$	4	$27000-25200-2760-60=-1020<0$

Therefore 3 is the firft Figure of the Root required; or it is fome Number between 30 and 40.

Example 2.

In the Equation	1	$a^3+2a^2-23a-70=0$, whose Limits are 1, 10
make $a=5$	2	$125+50-115-70=-10<0$
$a=6$	3	$216+72-138-70=+80>0$

Therefore the Root is between 5 and 6.

Example 3

In the Equation, or	1	$-a^4+80a^3-1998a^2+14937a-5000=0$, or $-a^4+80a^3-2000a^2+15000a-5000=0$ near, whose Limits are 10. 100
$a=50$	2	$-6250000+1000000-5000000+750000-5000=-9505000<0$
$a=20$	3	$-160000+640000-800000+300000-5000=-25000<0$
$a=10$	4	$-10000+80000-200000+150000-5000=+15000>0$

Therefore the Root required is some Number between 10 and 20.

Problem XIII.

502. To determine the first two Figures of the greatest Positive Root in a given Adfected Equation.

Effection.

1. Find the first Figure of the Root by the last, and call it b.
2. Put e for the remaining Part of the Root.
3. Substitute $b+e$ for the Root of the given Equation.
4. Reject all the Powers of e above the first.
5. Find e to the first or second Figure in the new Equation (In. 476.)
6. Add e so found to b, and the Sum will be nearly the first two or three Figures required.

Example 1.

Let it be required to find the first two Figures of the Root of the Equation $a^3+2a^2-23a-70=0$.

Here $b=5$ (In. 501.) or $5+e=a$, therefore

	1	$+a^3=125+75e+15e^2+e^3$
	2	$+2a^2=50+20e+2e^2$
	3	$-23a=-115-23e$
	4	$-70=-70$
1+2+3+4	5	$-10+72e+17e^2+e^3=0$
Pre. 4	6	$-10+72e=0$, or $10=72e$
6÷72	7	$\frac{10}{72}=e=0.13$, &c.
7+5	8	$5+0.13$, &c. $=5.13$, the three first Figures of the Root required.

Example

Example 2.

It is required to find the first two Figures of the greatest Positive Root in the Equation $-a^4+80a^3-1998a^2+14937-5000=0$. Here $b=10$, *i. e.* $10+e=a$; therefore

	1	$-a^4 = \quad -10000 \quad -4000e$ &c.
	2	$+80a^3 = \quad +80000 \quad +24000e$ &c.
	3	$-1998a^2 = \quad -199800 \quad -39960e$ &c.
	4	$+14937a = \quad +149370 \quad +14937e$
	5	$-5000 = \quad -5000$
$1+2+3+4+5$	6	$+14570 \quad -5023e$ &c. $=0$
or	7	$14570 = 5023e$
$7 \div \overline{5023}$	8	$\frac{14570}{5023} = e = 2.8$ &c.
$8+b$	9	$b+e=10+2.8=12.8$ the three first Figures of the required Root, nearly.

SCHOLIUM X.

503. And this is the same with Mr. *Ralphson*'s Method of Approximation in his *Universal Analysis*.

CHAP. VII.

Of the Resolution of Inadfected Compound Equations.

PROBLEM XIV.

504. TO resolve any Inadfected Compound Equation $a^m=r$; or, which is the same thing, to extract the m Root from any given Resolvend r.

Effection.

1. Distribute the given Resolvend into Periods (In. 28, 289.)
2. Seek the nearest homologous Power to the highest Period, which call b^m (In. 349, 350.) and note whether it be greater or lesser than just, whose Root b is the first Part of $r^{\frac{1}{m}}$ the Root required.

3. Put

3. Put $e =$ the remaining Part of the Root unknown, *i. e.* make $b+e=r^{\frac{1}{m}}$, if b be assumed lesser than just, and $b-e=r^{\frac{1}{m}}$ if greater. So that instead of $a^m=r$ we have $b^m \pm mb^{m-1}e+pb^{m-2}e^2 \pm qb^{m-3}e^3$ &c. $=r$ (In. 404.) according as b is taken greater or lesser than just.
4. Reject all the Powers of e above the Square by reason of their Smallness (In. 160, 161.) then

If b be taken lesser than just,

	1	$b^m+mb^{m-1}e+pb^{m-2}e^2=r=a^m$
$1-b^m$	2	$mb^{m-1}e+pb^{m-2}e^2=r-b^m=d$
$2 \div pb^{m-2}$	3	$\frac{m}{p}be+ee=\frac{d}{pb^{m-2}}$
$3 \div \overline{\frac{m}{p}b+e}$	4	$e=\dfrac{\frac{d}{pb^{m-2}}}{\frac{m}{p}b+e}$
$4+b$	5	$b+e=b+\dfrac{\frac{d}{pb^{m-2}}}{\frac{m}{p}b+e}=r^{\frac{1}{m}}=a$

If b be taken greater than just,

	1	$b^m-mb^{m-1}e+pb^{m-2}e^2=r=a^m$
	2	$mb^{m-1}e-pb^{m-2}e^2=b^m-r=d$ (In. 430.)
Whence	3	$b-e=b-\dfrac{\frac{d}{pb^{m-2}}}{\frac{m}{p}b-e}=r^{\frac{1}{m}}=a$

Therefore the Root a of any given Power $a^m=r$ will be found nearly equal (in the Square a^2 exactly equal) to

$$b \pm \dfrac{\frac{d}{pb^{m-2}}}{\frac{m}{p}b \pm e},$$ the Theorem (In. 351.)

But

But a yet more exact Method may be invented for extracting the Roots from all Powers above the Square by retaining the Cube of e in the Resolvend, as follows.

Case 1. with b less than just,

	1	$r = b^m + mb^{m-1}e + pb^{m-2}e^2 + qb^{m-3}e^3$
$r - b^m$	2	$d = r - b^m = mb^{m-1}e + pb^{m-2}e^2 + qb^{m-3}e^3$
Subst.	3	$d = mb^{m-1}e + pb^{m-2}e^2$ from the last.
$3 - mb^{m-1}e$	4	$d - mb^{m-1}e = pb^{m-2}e^2$
$4 \div pb^{m-2}$	5	$\frac{d - mb^{m-1}e}{pb^{m-2}} = e^2$
$5 \times qb^{m-3}e$	6	$\frac{qde - qmb^{m-1}e^2}{pb} = qb^{m-3}e^3$
2. 6.	7	$d = mb^{m-1}e + pb^{m-2}e^2 + \frac{qde - qmb^{m-1}e^2}{pb}$
$7 \times pb$	8	$pbd = pmb^m e + p^2 b^{m-1}e^2 + qde - qmb^{m-1}e^2$ $= pmb^m e + qde + \overline{p^2 - qm}\vert b^{m-1}ee$
$8 \div \overline{p^2 - qm}\vert b^{m-1}$	9	$\frac{pd}{\overline{p^2 - qm}\vert b^{m-2}} = \frac{pm}{p^2 - qm}be + \frac{qd}{\overline{p^2 - qm}\vert b^{m-1}}e + ee$
$9 \div \frac{pm}{p^2 - qm}b + \frac{qd}{p^2 - qmb^{m-1}} + e$	10	$\dfrac{\frac{pd}{p^2 - qmb^{m-2}}}{\frac{pm}{p^2 - qm}b + \frac{qd}{p^2 - qmb^{m-1}} + e} = e$
$10 + b$	11	$b + \dfrac{\frac{pd}{p^2 - qm\,b^{m-2}}}{\frac{pm}{p^2 - qm}b + \frac{qd}{p^2 - qmb^{m-1}} + e} = b + e = r^{\frac{1}{m}}$

Case 2. with b assumed greater than just.

	1	$r=b^m-mb^{m-1}e+pb^{m-2}e^2-qb^{m-3}e^3$
b^m-1	2	$b^m-r=d=mb^{m-1}e-pb^{m-2}e^2+qb^{m-3}e^3$
Substitute	3	$d=mb^{m-1}e-pb^{m-2}e^2$ from the last ∴
$mb^{m-1}e-3$	4	$mb^{m-1}e-d=pb^{m-2}e^2$
$4\div pb^{m-2}$	5	$\frac{mb^{m-1}e-d}{pb^{m-2}}=e^2$
$5\times qb^{m-3}e$	6	$\frac{qmb^{m-1}e^2-qde}{pb}=qb^{m-3}e^3$
2. 6.	7	$d=mb^{m-1}e-pb^{m-2}e^2+\frac{qmb^{m-1}e^2-qde}{pb}$.
$7\times pb$	8	$pmb^m e-p^2b^{m-1}e^2+qmb^{m-1}e^2-qde=pbd=$ $pmb^m e-qde-\overline{p^2-qm}\,b^{m-1}e^2$

Whence proceeding as above, we have

$$b-\frac{\dfrac{pd}{\overline{p^2-qm}|\,b^{m-2}}}{\dfrac{pm}{p^2-qm}\,b-\dfrac{qd}{\overline{p^2-qm}|\,b^{m-1}}-e}=b-e=r^{\frac{1}{m}}$$

Scholium XI.

505. The former of these Theorems is the same with the Irrational Formula of M. *de Lagney* (who follow'd the ingenious *Joseph Ralphson, F.R.S.*) and is applied by the celebrated Dr. *Edmund Halley* to the Resolution of all sorts of Adfected Equations, as shall be shewn (In. 509.)

CHAP. VIII.

Of the Resolution of Adfected Equations.

Problem XV.

506. TO resolve an Adfected Quadratic Equation by compleating the Square.

Effection

Effection.

1. Bring the Abſolute Number to one ſide of the Equation (In. 432.)
2. Add $\frac{1}{4}$ the Square of the Coefficient of the ſecond Term to both ſides of the Equation.
3. Extract the Square Root from both ſides.
4. Add or ſubtract one half the ſaid Coefficient to or from both ſides, and it is done.

Example 1.

Reſolve the Equation $a^2+4a-77=0$, or $a^2+4a=77$ into its conſtituent Roots, which are one Poſitive, and the other Defective.

	1	$a^2+4a=77$
	2	$+4=+4$
$1+\overline{2}^2$	3	$a^2+4a+4=81$ (In. 406.)
$3w^2$	4	$a+2=\overline{81}^{\frac{1}{2}}=9$
$4-2$	5	$a=9-2=7$, the Poſitive Root.
∴	6	$\frac{a^2+4a-77=0}{a-7=0}=a+11=0$, *i. e.* $a=-11$ the Defective Root (In. 511.)

Example 2.

Reſolve $a^2-23a-374=0$, or $a^2-23a=374$.

	1	$a^2-23a=374$
	2	$+\frac{529}{4}=+\frac{529}{4}$
$1+\frac{23^2}{2^2}$	3	$a^2-23a+\frac{529}{4}=\frac{2025}{4}$
$3w^2$	4	$a-\frac{23}{2}=\frac{45}{2}$
$4+\frac{23}{2}$	5	$a=\frac{68}{2}=34$ the Poſitive Root.
∴	6	$\frac{a^2-23a-374=0}{a-34=0}=a+11=0$, *i. e.* $a=-11$, the Defective Root.

Example

Case 2. with b assumed

$b^m - 1$

Substitute

$mb^{m-1}\ e - 3$

$4 \div pb^{m-2}$

$5 \times qb^{m-7}$

2.

$7 \times p$

Example 3.

Resolve $-a^2+32a-255=0$, *or* $a^2-32a+255=0$.
Here the Roots are both Positive

$1 \mid a^2-32a=-255$
$+256=+156$
$2 \mid a^2-32a+256=1$
$3 \mid a-16=1$
$4 \mid a=+17$ *the one Root.*
$5 \mid -a^2+32a-255=0$
$6 \mid a=\ a-17=0$

$1+2$, $3w^2$, $4+16$

$= -a+15=0$, *i. e.* $a=15$ the other Positive Root.

Example 4.

Resolve $a^2 - a + \frac{2l}{d}a - \frac{2s}{d} = 0$

$$1 \mid a^2 - \overline{1-\frac{2l}{d}}\, a = \frac{2s}{d} = \frac{8sd}{4dd}$$

$$1+\frac{d-2l}{d} \quad 2 \mid a^2 - \overline{1-\frac{2l}{d}}\, a + \frac{dd-4dl+4ll}{4dd} = \frac{8sd+dd-4dl+4ll}{4dd}$$

$$2w^2 \quad 3 \mid a - \frac{a-2l}{2d} = \left|\frac{8sd+dd-4dl+4ll}{4dd}\right|^{\frac{1}{2}} = \left|\frac{8sd+dd-4dl+4ll}{2d}\right|^{\frac{1}{2}}$$

$$3+\frac{d-2l}{2d} \quad 4 \mid a = \frac{\overline{8ds+dd-4dl+4ll}\,^{\frac{1}{2}} + d - 2l}{2d}$$

Problem XVI.

507. To extract the Root from an Adfected Cubic Equation by Approximation, according to In. 354.

Example 1.

Let it be required to extract the Root from the Adfected Cubic Equation $a^3-28.25a^2-91.75a-62.5=0$; which consists of one Positive and two Defective Roots (In. 465.) And the Positive Root is some Number between 30 and 40 (In. 501.) therefore make $b=30$ less than just, *i. e.* $30+e$ or $b+e=a$

$$\begin{array}{rl}
+a^3 = & +27000 + 2700e + 9e^2 + e \\
-28.25a^2 = & -25425 - 1695e - 28.25e^2 \\
-91.75a = & -2752.5 - 91.75e \\
-62.5 = & -62.5 \\
\hline
& -1240 + 913.25e + 61.75e^2 + e^3 = 0
\end{array}$$

Then

Then putting $d=1240$, $s=913.25$, $t=61.75$, we have the Equation $-d+se+tee+eee=0$, or,

	1	$se+tee+eee=d$
Or	2	$se+tee=d$ rejecting e^3
$2-se$	3	$tee=d-se$
$3\div t$	4	$ee=\dfrac{d-se}{t}$
$4\times e$	5	$eee=\dfrac{de-see}{t}$
$2+5=1$	6	$se+tee+\dfrac{de-see}{t}=d$
$6\times t$	7	$tse+tte+de-see=td$, or
$6\times t$	8	$\overline{ts+d}\,\vert e+\overline{tt-s}\,\vert ee=td$
$8\div\overline{tt-s}$	9	$\dfrac{ts+d}{tt-s}e+ee=\dfrac{td}{tt-s}$
$9\div\dfrac{ts+d}{tt-s}+e$	10	$e=\dfrac{\dfrac{td}{tt-s}}{\dfrac{ts+d}{tt-s}+e}=\dfrac{26.4051}{19.8727+e}=1.25007$

Whence $b+e=31.25007=a$ true to the seventh Figure at the first Operation, or $a=31.25$ just; then $\dfrac{a^3-28.25a^2-91.75a-62.5=0}{a-31.25=0}=a^2+3a+2=0$ And the two Roots of $a^2+3a+2=0$ are -1 and -2. Therefore all the Roots of the given Equation are $+31.25$, -1 and -2.

Example 2.

Resolve the Equation $a^3-6a^2+24a-20039=0$.

The greatest Positive Root in this Equation is between 20 and 30 (In. 501.) but nearer the latter, therefore assume $30=b$ more than just, *i. e.* $b-e=a$

$$
\begin{array}{rl}
+a^3= & +27000-2700e+90ee-eee \\
-6a^2= & -5400+360e-6ee \\
+24a= & +720-24e \\
-20039= & -20029 \\
\hline
 & 2281-2364e+84ee-e^3=0 \\
\textit{i. e.} & d-se+tee-eee=0
\end{array}
$$

	1	$se-te^2+e^3=d$
	2	$se-tee=d$ rejecting eee
$2+tee-d$	3	$se-d=tee$
$3\div t$	4	$\frac{se-d}{t}=ee$
$4\times e$	5	$\frac{see-de}{t}=eee$
$2+5$	6	$se-te^2+\frac{see-de}{t}=d$
$6\times t$	7	$ste-ttee+see-de=td$ or
	8	$\overline{st-d}\lvert e-\overline{tt-s}\rvert ee=td$
$8\div\overline{tt-s}$	9	$\frac{st-d}{tt-s}e-ee=\frac{td}{tt-s}$
$9\div\left\lvert\frac{st-d}{tt-s}-e\right.$	10	$e=\frac{\frac{td}{tt-s}}{\frac{st-d}{tt-s}-e}=\frac{40.836317\ \&c.}{41.836104\ \&c.-e}=1.000006\ \&c.$
$b-10$	11	$b-e=28.999994=a$ true to the eighth Figure at the

first Operation, or $a=29$ exact for the first Root Positive. Then

$$\frac{a^3-6a^2+24a-20039=0}{a-29=0}=a^2+23a+691=0,\ \text{whence}$$

	1	$a^2+23a=-691$
$1+\frac{\overline{23}^2}{4}$	2	$a^2+23a+132.25=+132.25-691=-558.75$
$2ɯ^2$	3	$a+11.5=+\overline{\lvert-558.75}^{\frac{1}{2}}$ which is impossible.
$3-\overline{11.5}$	4	$a=+\overline{\lvert-558.75}^{\frac{1}{2}}-11.5$ the second Root

which is impossible (In. 470.) And the Reason you see is, because the Square of one half the Coefficient of the second Term, *i. e.* the Square of $\frac{23}{2}$ is less than the Absolute Number 691.

Lastly, $\frac{a^2+23a+691}{a-\lvert-558.75+11.5}=a+\overline{\lvert-5}^{\frac{1}{2}}+1.$ Whence $a=-\overline{\lvert-5}^{\frac{1}{2}}-1$ the third Root impossible.

Scholium XII.

508. In the foregoing Examples I have purposely omitted assuming any more than the first Figure of the Root; but if more can be assumed, the first

first Operation will seldom or never miss of bringing forth quintuple the Figures at least, but generally more, as may be seen. But this Method fails in all Adfected Equations of above three Dimensions: And therefore for the Resolution of these, we must have Recourse to the following Method of Dr. *Hally*, which, I believe, by far exceeds all the Methods that ever have been hitherto invented for the like Purpose.

PROBLEM XVII.

509. To extract the Roots from all kind of Equations by Approximation.

Example 1.

Let it be required to resolve the Equation $-a^3+330a^2-1600a-8125=0$. The greatest Positive Root of this Equation is some Number between 320 and 330 (In. 502.) therefore assume $b=320$ less than just, *i. e.* $320+e=a$

$$\begin{aligned}
-a^3 &= -32768000-307200e-960ee-eee\\
+330aa &= +33792000+211200e+330ee\\
-1600a &= -\quad 512000-\quad 1600e\\
-8125 &= -\qquad 8125\\
\hline
& +503875-\ 97600e-630ee-eee=0\\
\textit{i. e.}\quad & \quad d\ -\quad se\ -\ tee\ -ee=0
\end{aligned}$$

	1	$se+tee=d-eee$
$1\div t$	2	$\frac{s}{t}e+ee=\frac{d}{t}-\frac{eee}{t}$
$2\div\left|\frac{s}{t}+e\right.$	3	$e=\dfrac{\frac{d}{t}}{\frac{s}{t}+e}-\dfrac{\frac{eee}{t}}{\frac{s}{t}+e}$

Then rejecting *eee* upon account of its Smallness

$$e=\frac{\frac{d}{t}}{\frac{s}{t}+e}=\frac{799.80158}{154.92063+e}=5.00119,$$ from which if we subtract

$$\frac{\frac{eee}{t}}{\frac{s}{t}+e}=\frac{0.1985}{164.9+e}=0.00120$$ (Step. 3.) we shall have $e=$

4.99999

4.99999 or rather $e=5$ exact; whence $b+e=a=325$ just, which is found by this Method true to the eighth Figure at the first Operation, by means of the Correction $\dfrac{\frac{eee}{t}}{\frac{s}{t}+e}-0$. The other Roots are $+8.09017$, &c. and -3.09017.

Example 2.

Let it be required to extract the Root from the Equation.

$$a^4-80a^3+1998a^2-14937a+5000=0 \text{ or}$$
$$-a^4+80a^3-1998a^2+14937a-5000=0$$

The first three Figures of this Equation are 12.7 (ln. 502.)

$$-a^4=-26014.4641-8193.532e-967.74ee-50.8eee-eeee$$
$$+80a^3=+163870.64+38709.600e+3048.00ee+80.0eee.$$
$$-1998a^2=-322257.42-50749.200e-1998.00ee$$
$$+14937a=+189699.9+14937.000e$$
$$-5000=-5000.$$

$$+ \quad 298.6559 \quad -5296.132e \quad +82.26ee+29.2eee-eeee$$
$$d \quad - \quad se \quad + \quad tee+ \quad veee-eeee$$

	1	$se-te^2-ve^3+e^4=d$
$1+ve^3-e^4$	2	$se-te^2=d+ve^3-e^4$
$2\div t$	3	$\frac{s}{t}e-ee=\frac{d}{t}=\frac{ve^3-e^4}{t}$
$3\div\frac{s}{t}-e$	4	$e=\dfrac{\frac{d}{t}}{\frac{s}{t}-e}+\dfrac{\frac{ve^3-e^4}{t}}{\frac{s}{t}-e}=\dfrac{\frac{d+ve^3-e^4}{t}}{\frac{s}{t}-e}$

Then rejecting all the Powers of e above ee

$$e=\frac{\frac{d}{t}}{\frac{s}{t}-e}=\frac{3.63063335764}{64.382834913-e}=0.0564408033\text{1 \&c.}$$

which

which being corrected by the Addition of $\frac{\frac{veee-eeee}{t}}{\frac{s}{t}-e}$ (Step 4.) =

$\frac{0.00006370266}{64.2699}=0.00000099117$ becomes 0.05644179448 &c. Whence $a=b+e=12.75644179448$ &c. true to the fourteenth Figure at the first Operation, by help of the Correction $\frac{\frac{veee-eeee}{t}}{\frac{s}{t}-e}$. And the Work will be yet farther corrected, if the last Value of e be substituted in the Equation $e=\frac{\frac{d+veee-eeee}{t}}{\frac{s}{t}-e}$ (Step. 4.) $=0.05644179448074402$, which gives $a=12.75644179448074402$ &c. exact to the twentieth Figure. And if more Figures be yet desired, let the last Value of e be again substituted in the same Equation, and that again in the same, by which Means the Root may be carried to any assigned Exactness, without varying the Coefficients of e.

Scholium XIII.

510. In the Effection of the last Problem, note,

1. That every Repetition of the Calculus, at least, triples the Figures assumed (In. 351.)
2. That the Corrections are of no Use, but when the first two Figures, of the Root, or a Number nearly equal to the first two Figures, is assumed; and the nearer the assumed Number is to just, still the more just will be the Corrections.
3. That the Divisor $\frac{s}{t}\pm e$ in the first Correction must always be the same with the latter Divisor, which is used in finding the first Value of e

Problem XVIII.

511. To resolve an Equation with the first Term multiplied into a known Quantity.

Ex. gr. Suppose the Biquadratic Equation $-2018aaaa+125409a^3-2464230.25a^2+354683307a-274183922.25=0$. *Ward*'s Introd. *p.* 336.

The Root of this Equation is some Number between 10 and 20: For if the Equation be divided by 2018, the Result will be $-a^4+62.1$ &c $a^3-1221.1$ &c $a^2+17575.9$ &c $a-135869.1$ &c $=0$. Whence are found the Limits 10, 160. Make $b=10$ *i. e.* $10+e=a$.

$$\begin{array}{lll}
\text{Then } -2018a^4 & = & -20180000-80720000e-12108000ee \text{ \&c.} \\
+125409a^3 & = & +125409000+37622700e+3762270ee \text{ \&c.} \\
-2464230.25a^2 & = & -246423025-49284605e-2464230.25ee \\
+35468307\,a & = & +354683070+35468307e \\
-274183922.25 & = & -274183922.25
\end{array}$$

$$\begin{array}{l}
\text{Whose Sum}-60694877.25+157344402e+87239.75ee \text{ \&c.} =0 \\
\qquad\qquad -\quad d \quad + \quad se \quad + \quad tee \quad =0
\end{array}$$

Or	1	$d=se+tee$
$1\div t$	2	$\frac{d}{t}=\frac{s}{t}e+ee$
$2\div\sqrt{\frac{s}{t}+e}$	3	$\dfrac{\frac{d}{t}}{\frac{s}{t}+e}=e=3.7$ &c.
$b+e$	4	$10+e=a=13.7$ &c. the first three Figures of the Root required.

Whence, and from the Principles already delivered, I presume the Learner will easily perceive how to carry the Root to any assigned Place.

PROBLEM XIX.

512. To extract the Root from a Compound Irrational Quantity.

Example

Example 1

Let it be required to extract the Square Root from the Equation $aa = b + \overline{4bcc - 4cccc}^{\frac{1}{2}}$ or to find the Value of $a = \overline{b + \overline{4bc^2 - 4c^4}^{\frac{1}{2}}}^{\frac{1}{2}}$

Put	1	$a = \overline{b + \overline{4bcc - 4cccc}^{\frac{1}{2}}}^{\frac{1}{2}} = e + y^{\frac{1}{2}}$
$1 \odot^{2}$	2	$a^2 = b + \overline{4bc^2 - 4cccc}^{\frac{1}{2}} = e^2 + 2e\overline{y}^{\frac{1}{2}} + y$
make	3	$e^2 + y = b$
make	4	$2e\overline{y}^{\frac{1}{2}} = \overline{4bc^2 - 4cccc}^{\frac{1}{2}}$
$4 \odot^{2}$	5	$4eey = 4bc^2 - 4cccc$
$3 \odot^{2}$	6	$e^4 + 2e^2y + y^2 = bb$
$6 - 5$	7	$e^4 - 2e^2y + y^2 = b^2 - 4bc^2 + 4cccc$
$7\omega^{2}$	8	$e^2 - y \qquad = b - 2c^2$
$8 + y$	9	$e^2 = b - 2c^2 + y$
$3 - y$	10	$e^2 = b - y$
9 · 10	11	$b - 2cc + y = b - y$, or $-2cc + y = -y$
∴	12	$2y = 2cc$
$12 \div 2$	13	$y = cc$
$13\omega^{2}$	14	$y^{\frac{1}{2}} = c$
9, 13,	15	$e^2 = b - 2c^2 + c^2 = b - c^2$, because $c^2 = y$
$15\omega^{2}$	16	$e = \overline{b - cc}^{\frac{1}{2}}$
$16 + 14$	17	$e + y^{\frac{1}{2}} = \overline{b - cc}^{\frac{1}{2}} + c = a$ (Step. 1.)
∴	18	a or $\overline{b + \overline{4bcc - 4cccc}^{\frac{1}{2}}}^{\frac{1}{2}} = \overline{b - cc}^{\frac{1}{2}} + c$

Example

Example 2.

Let be required to extract the Square Root from the Equation $a^2 = b - \overline{4bcc - 4cccc}|^{\frac{1}{2}}$, or which is the same, from $a^2 = b - 2c\,\overline{b - cc}|^{\frac{1}{2}}$

Put	1	$a = \overline{b - 2c\,\overline{b-cc}\|^{\frac{1}{2}}}\|^{\frac{1}{2}} = e - y^{\frac{1}{2}}$
$1 \odot^2$	2	$a^2 = b - 2c\,\overline{b-cc}\|^{\frac{1}{2}} = e^2 - 2e\,\overline{y}\|^{\frac{1}{2}} + y$
make	3	$e^2 + y = b$
make	4	$2e\,\overline{y}\|^{\frac{1}{2}} = 2c\,\overline{b-cc}\|^{\frac{1}{2}}$
$3 \odot^2$	5	$e^4 + 2e^2y + y^2 = b^2$
$4 \odot^2$	6	$4e^2y = 4ccb - 4cccc$
5 — 6	7	$e^4 - 2e^2y + y^2 = b^2 - 4ccb - 4cccc$
$7 \omega^2$	8	$e^2 - y = b - 2cc$
$8 + y$	9	$e^2 = b - 2cc + y$
$3 - y$	10	$e^2 = b - y$
9. 10	11	$b - 2cc + y = b - y$, or $-2cc + y = -y$
	12	$2y = 2cc$
$12 \div 2$	13	$y = cc$
$13 \omega^2$	14	$y^{\frac{1}{2}} = c$
9, 13	15	$e^2 = b - 2c^2 + c^2 = b - cc$
$15 \omega^2$	16	$e = \overline{b-cc}\|^{\frac{1}{2}}$
$16 + 14$	17	$e - y^{\frac{1}{2}} = \overline{b-cc}\|^{\frac{1}{2}} - c = a$

Example 3.

Let it be required to extract the Cube Root from the Equation $a^3 = 20 + \overline{392}|^{\frac{1}{2}}$, or bring $a = \overline{20 + \overline{392}|^{\frac{1}{2}}}|^{\frac{1}{3}}$ to a more simple Expression.

Put

Put	1	$a = \overline{20+\overline{392}^{\frac{1}{2}}}^{\frac{1}{3}} = e+y^{\frac{1}{2}}$
$1 \odot^{3}$	2	$aaa = 20+\overline{392}^{\frac{1}{2}} = e^3+3e^2\,\overline{y}^{\frac{1}{2}}+3ey+y^{\frac{3}{2}}$
Make	3	$e^3+3ey=20$
Make	4	$3e^2\,\overline{y}^{\frac{1}{2}}+y^{\frac{3}{2}} = \overline{392}^{\frac{1}{2}}$
$3 \odot^{2}$	5	$e^6+6e^4y+9e^2y^2=20^2=400$
$4 \odot^{2}$	6	$9e^4y+6e^2y^2+y^3=392$
$5-6$	7	$e^6-3e^4y+3e^2y^2-y^3=400-392=8$
$7 w^3$	8	$e^2-y=2$
∴	9	$e^2-2=y$
3, 9,	10	$e^3+3e^3-6e=4e^3-6e=20$
$10 \div \overline{4}$	11	$e^3+0e^2-\frac{6}{4}e=\frac{20}{4}$
Subst. $2e=u$	12	$\frac{1 \quad 2 \quad 4 \quad 8}{u^3+0u^2-6u=40}$ (In. 487.)
$12 w^3$	13	$u=4$
12, 13.	14	$e=\frac{u}{2}=2$
9, 14.	15	$y^2=e^2-2=2$
$15 w^2$	16	$y^{\frac{1}{2}}=2^{\frac{1}{2}}$
$14+16$	17	$e+y^{\frac{1}{2}}=2+2^{\frac{1}{2}}=\overline{20+\overline{392}^{\frac{1}{2}}}^{\frac{1}{3}}$ the Root required (Step 1.)

SCHOLIUM XIV.

513. To which may be referred what Sir *Isaac Newton* teaches concerning the Reduction of Radical Quantities to more simple Radicals. Page 51 of his Book of *Universal Arithmetic.*

PROBLEM XX.

514. To extract the Root y from the Infinitinomial $Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6$ &c. $=r$.

Effection.

1. For y substitute $Hr+Ir^2+Kr^3+Lr^4+Mr^5+Nr^6$ &c. then by In. 413.

$y^2=$

$$\begin{array}{ll}
y^2 = H^2r^2 + 2HIr^3 + \begin{matrix}2HK\\II\end{matrix}\, r^4 + \begin{matrix}2HL\\2IK\end{matrix}\, r^5 + \begin{matrix}2HM\\2IL\\KK\end{matrix}\, r^6 \text{ \&c.} \\[2ex]
y^3 = \qquad\quad H^3r^3 + 3H^2I\, r^4 + \begin{matrix}3H^2K\\3HI^2\end{matrix}\, r^5 + \begin{matrix}3H^2L\\6HIK\\III\end{matrix}\, r^6 \text{ \&c.} \\[2ex]
y^4 = \qquad\qquad\qquad H^4\, r^4 + 4H^3I\, r^5 + \begin{matrix}6H^2I^2\\4H^3K\end{matrix}\, r^6 \text{ \&c.} \\[2ex]
y^5 = \qquad\qquad\qquad\qquad\quad H^5\, r^5 + 5H^4I\, r^6 \text{ \&c.} \\[1ex]
y^6 = \qquad\qquad\qquad\qquad\qquad\qquad H^6\, r^6 \text{ \&c.} \\[1ex]
\text{\&c.} \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{\&c.}
\end{array}$$

2. Substitute these Values of y in the Equation $Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6$ &c. $=r$, or $-r+Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6$ &c. $=0$; thus,

$$\begin{array}{llllll l}
-r=-1 & & & & & & \\
+Ay=+AH & +AI & +AK & +AL & +AM & +AN & \\
 & & & & & +2BHM & \\
+By^2= & +BH^2 & +2BHI & +2BHK & +2BHL & +2BIL & \\
 & & & +BII & +2BIK & +BKK & \\
 & & & & & +3CH^2L & \\
+Cy^3= & & +CH^3 & +3CH^2I & +3CH^2K & +6CHIK & \\
 & & & & +3CHI^2 & +CIII & \\
+Dy^4= & & & +DH^4 & +4DH^3I & +6DH^2I^2 & \\
 & & & & & +4DH^3K & \\
+Ey^5= & & & & +EH^5 & +5EH^4I & \\
+Fy^6= & & & & & +FH^6 & \\
\text{\&c.} & & & & & \text{\&c.} & \\
r & r^2 & r^3 & r^4 & r^5 & r^6 \text{ \&c.} & =0
\end{array}$$

3. Divide the last Equation by r (ln. 486.) and make each of the Coefficients of r r^2, r^3, r^4 &c. equal to nothing, (because the Whole is so) as follows.

1st Coefficient	1	$AH-1=0$ or $AH=1$
whence	2	$H=\frac{1}{A}$
2d Coefficient	3	$AI+BH^2=0$ or $AI=-BHH$
whence	4	$I=-\frac{B}{A^3}$, substituting $\frac{1}{A}$ for H (Step. 2.)
3d Coefficient	5	$AK+2BHI+CH^3=0$
whence	6	$K=\frac{2B^2-AC}{A^5}$, substituting $\frac{1}{A}$ for H, and $-\frac{B}{A^3}$ for I.
4th Coefficient	7	$AL+2BHK+BI^2+3CH^2I+DH^4=0$
whence	8	$L=\frac{5ABC-A^2D-5B^3}{A^7}$ substituting $\frac{2B^2-AC}{A^5}=K$.
5th Coefficient	9	$AM+2BHL+2BIK+3CH^2K+3CHI^2+DH^3I+4EH^5=0$
whence	10	$M=\frac{14B^4-21AB^2C+6A^2BD+3H^2C^2-A^3E}{A^9}$
6th Coefficient	11	$+AN+2BHM+2BIL+BKK+3GH^2L+6CHIK+CI^3$ $+6DH^2I^2+4DH^3K+5EH^4I+FH^6=0$
whence	12	$N=\frac{-42B^5+84AB^3C-28A^2B^2D-28A\cdot BC^2+7A^3BE+7A^3DC-A^4F}{A^{11}}$

4. Substitute these Values of H, I, K, L, M, N, &c. in the Equation $Hr+Ir^2+Kr^3+Lr^4+Mr^5+Nr^6$ &c. $=y$ and you will have the Root sought,
$$y=\frac{1}{A}-\frac{B}{A^3}r^2+\frac{2B^2-M}{A^5}r^3+\frac{5ABC-A^2D-5B^3}{A^7}r^4+\frac{14B^4-21AB^2C+6A^2BD+3A^2C^2-A^3E}{A^9}r$$ &c. *ad infinitum.*

Scholium XV.

515. The ingenious Mr. *Abr. De Moivre*, is the Author of the Effection of the last Problem.

CHAP.

CHAP. IX.

Of the Reduction of Equations of four, six, eight, ten, &c. Dimensions, by Surd Divisors: Or the Method of dividing Equations of this Sort into two such equal Parts that the Root may be extracted out of each. From Sir Isaac Newton.

PROBLEM XXI.

516. TO reduce an Equation of four Dimensions.

Effection.

Suppose $x^4+px^3+qxx+rx+s=0$, where p, q, r, and s denote the known Quanties of the Terms of the Equation affected with their proper Signs. Make

$$q-\tfrac{1}{4}pp=\alpha. \quad r-\tfrac{1}{2}\alpha p=\beta.$$
$$s-\tfrac{1}{4}\alpha\alpha=\zeta.$$

Then put for n some common Integral Divisor of the Terms β and 2ζ, that is not a Square, and which ought to be odd, and divided by 4 to leave Unity, if either of the Terms p and r be odd. Put also for k some Divisor of the Quantity $\frac{\beta}{n}$ if p be even; or half of the odd Divisor, if p be odd; or nothing, if the Dividual β be nothing. Take the Quotient from $\frac{1}{2}pk$, and call the half of the Remander l. Then for Q put $\frac{\alpha+nkk}{2}$, and try if n divides $QQ-s$, and the Root of the Quotient be rational and equal to l; which if it happen, add to each Part of the Equation $nkkxx+2nklx+nll$, and extract the Root on both Sides, there coming out $xx+\frac{1}{2}px+Q=n^{\frac{1}{2}}$ into $\overline{kx+l}$.

Ex. gr. Suppose the Equation $x^4-2ax^3\begin{smallmatrix}+2aa\\-cc\end{smallmatrix}xx-2a^3x+a^2=0$. By substituting $-2a$, $2aa-cc$, $-2a^3$ and $+a^4$ for p, q, r, and s, respectively, you obtain $aa-cc=\alpha$, $-acc-a^3=\beta$, and $\frac{1}{4}a^4+\frac{1}{2}aacc-\frac{1}{4}c^4=\zeta$. The common Divisor of the Quantities β and 2ζ is $aa+cc$, which then will be n; and $\frac{\beta}{n}$ or $-a$, has the Divisors 1 and a. But be-

cause

cause n is of two Dimensions, and $k\overline{|l}^{\frac{1}{2}}$ ought to be of no more than one, therefore k will be of none, and consequently cannot be a. Let therefore $k = 1$, and $\frac{\beta}{n}$ being divided by k, take the Quotient $-a$ from $\frac{1}{2}pk$ or $-a$, and there will remain nothing for l. Moreover $\frac{\alpha + nkk}{2}$, or aa is Q, and $QQ - s$, or $a^4 - a^4$, is o; and thence again there comes out nothing for l. Which shews the Quantities n, k, l, and Q, to be rightly found; and adding to each Part of the Equation propos'd the Terms $nkkxx + 2nklx + nll$, that is $aaxx + ccxx$, that the Root may be extracted on both Sides; and by that Extraction there will come out $xx + \frac{1}{2}px + Q = n^{\frac{1}{2}} \times \overline{kx + l}$, that is, $xx - ax + aa = \mp x\,\overline{|aa + cc}^{\frac{1}{2}}$. And the Root being again extracted, you'll have $x = \frac{1}{2}a \pm \frac{1}{2}\overline{aa + cc}^{\frac{1}{2}}$ or $\pm\, \overline{\frac{1}{4}cc - \frac{1}{2}aa \pm \frac{1}{2}a \times \overline{aa + cc}^{\frac{1}{2}}}^{\frac{1}{2}}$ See *Newton*'s Algebra *p.* 214.

Problem XXII.

517. To reduce an Equation of six Dimensions.

Effection.

Suppose the Equation $x^6 + px^5 + qx^4 + rx^3 + sxx + tx + v = 0$, and make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}p\alpha = \beta. \quad s - \tfrac{1}{2}p\beta = \gamma.$$
$$\gamma - \tfrac{1}{4}\alpha\alpha = \zeta. \quad t - \tfrac{1}{2}\alpha\beta = \eta. \quad v - \tfrac{1}{4}\beta\beta = \theta.$$
$$\zeta\theta - \tfrac{1}{4}\eta\eta = \lambda.$$

Then take for n out of the Terms 2ζ, η, 2θ, some common Integer Divisor, that is not a Square, and that likewise is not divisible by a Square Number, and which also divided by the Number 4 shall leave Unity; if but any one of the Terms p, r, t be odd. For k take some Integer Divisor of the Quantity $\frac{\lambda}{2nn}$ if p be even; or the half of an odd Divisor if p be odd; or o if λ be o. For Q [take] the Quantity $\frac{1}{2}\alpha + \frac{1}{2}nkk$. For l some Divisor of the Quantity $\frac{Qr - QQp - t}{n}$ if Q be an Integer; or the half of an odd Divisor, if Q be a Fraction that has for its Denominator the Number 2; or o, if the Dividual [or the Quantity] $\frac{Qr - QQp - t}{n}$ be nothing. And for R the Quantity $\frac{1}{2}r - \frac{1}{2}Qp + nkl$. Then try if $RR - v$ can be divided by n, and the Root of the Quotient extracted; and besides, if that Root be equal as well to the

O Quantity

Quantity $\frac{QR-\frac{1}{2}t}{nl}$ as to the Quantity $\frac{QQ+pR-nll-s}{2nk}$. If all these happen, call the Root m; and in room of the Equation propos'd, write thus, $x^3+\frac{1}{2}pxx+Qx+R=\pm n^{\frac{1}{2}}\times\overline{kxx+lx+m}$. For this Equation, by squaring its Parts, and taking from both Sides the Terms on the Right-Hand, will produce the Equation propos'd. But if all these Things do not happen in the Case propos'd, the Reduction will be impossible, if it appears beforehand that the Equation cannot be reduc'd by a rational Divisor.

For Example, let there be propos'd the Equation

$$x^6-2ax^5+2bbx^4+2abbx^3\begin{array}{l}-2aabb\\+2a^3b\\-4ab^3\end{array}xx\begin{array}{l}+3aab^4\\-a^4bb\end{array}=0.$$

and by writing $-2a$, $+2bb$, $+2abb$, $-2aabb+2a^3b-4ab^3$, 0, and $3aab^4-a^4bb$ for p, q, r, s, t, and v respectively, there will come out $2bb-aa=\alpha$. $4abb-a^3=\beta$. $2a^3b+2aabb-4ab^3-a^4=\gamma$. $-b^4+2a^3b+3aabb-4ab^3-\frac{1}{4}a^4=\zeta$. $-\frac{1}{2}a^5+3a^3bb-4ab=\eta$, and $-aab^4+a^4bb-\frac{1}{4}a^6=\theta$. And the common Divisor of the Terms 2ζ, η, and 2θ, is $aa-2bb$, or $2bb-aa$, according as aa or $2bb$ is the greater. But let aa be greater than $2bb$, and $aa-2bb$ will be n. For n must always be Affirmative. Moreover, $\frac{\zeta}{n}$ is $-\frac{1}{4}aa+2ab+\frac{1}{2}b$, $\frac{\eta}{n}$ is $\frac{1}{2}a^3+2ab^2$, and $\frac{\theta}{n}$ is $-\frac{1}{4}a^4+\frac{1}{2}aabb$, and consequently $\frac{\zeta}{2n}\times\frac{\theta}{n}-\frac{\eta\eta}{8nn}$ or $\frac{\lambda}{2nn}$, is $\frac{1}{8}a^6-\frac{1}{4}a^5b-\frac{3}{8}a^4bb+\frac{1}{2}a^3b^3-\frac{1}{8}aab^4$, the Divisors whereof are 1, a, aa; but because $n^{\frac{1}{2}}\times k$ cannot be of more than one Dimension, and $\overline{n}^{\frac{1}{2}}$ is of one, therefore k will be of none; and consequently can only be a Number. Wherefore, rejecting a and aa, there remains only 1 for k. Besides, $\frac{1}{2}\alpha+\frac{1}{2}nkk$ gives 0 for Q, and $\frac{Qr-QQp-t}{n}$ is also nothing; and consequently l, which ought to be its Divisor, will be nothing. Lastly, $\frac{1}{2}r-\frac{1}{2}pQ+nkl$ gives abb for R. And $RR-v$ is $-2aab^4+a^4bb$, which may be divided by n, or $aa-2bb$, and the Root of the Quotient $aabb$ be extracted, and that Root taken negatively, *viz.* $-ab$, is not unequal to the indefinite Quantity $\frac{QR-\frac{1}{2}t}{nl}$, or $\frac{0}{0}$, but equal to the definite Quantity $\frac{QQ+pR-nll-s}{2nk}$. Wherefore that Root $-ab$ will be m, and in the room of the Equation propos'd, there may be writ $x^3+\frac{1}{2}pxx+Qx+R=n^{\frac{1}{2}}\times\overline{kxx+lx+m}$, that is, $x^3-axx+abb=\overline{aa-2bb}^{\frac{1}{2}}\times\overline{xx-ab}$ The Truth of which

which Conclusion you may prove by squaring the Parts of the Equation found, and taking away the Terms on the Right Hand from both Sides. For from that Operation will be produc'd the Equation $x^6 - 2ax^5 + 2bbx^4 + 2abbx^3 - 2aabbxx + 2a^3bxx - 4ab^3xx + 3aab^4 - a^4bb = 0$, which was to be reduc'd.

Problem XXIII.

518. To reduce an Equation of eight, ten, twelve, &c. Dimensions.

Effection.

If the Equation is of eight Dimensions, let it be $x^8 + px^7 + qx^6 + rx^5 + sx^4 + tx^3 + vxx + wx + z = 0$, and make $q - \frac{1}{4}pp = \alpha$. $r - \frac{1}{2}p\alpha = \beta$. $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$. $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$. $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \epsilon$. $w - \frac{1}{2}\beta\gamma = \zeta$, and $z - \frac{1}{4}\gamma\gamma = \eta$. And seek a common Divisor of the Terms 2δ, 2ϵ, 2ζ, 8η, that shall be an Integer, and neither a Square Number; nor divisible by a Square Number; and which also divided by 4 shall leave Unity, if any of the alternate Terms p, r, t, w be odd, if there be no such common Divisor, it is certain, that the Equation cannot be reduc'd by the Extraction of a Quadratic Surd Root, and if it cannot be so reduc'd, there will scarce be found a common Divisor of all those four Quantities. The Operation therefore hitherto is a Sort of an Examination, whether the Equation be reducible or not; and consequently, since that Sort of Reductions are seldom possible, it will most commonly end the Work.

And, by a like Reason, if the Equation be of ten, twelve, or more Dimensions, the Impossibility of its Reduction may be known. As if it be $x^{10} + px^9 + qx^8 + rx^7 + sx^6 + tx^5 + vx^4 + ax^3 + bx^2 + cx + d = 0$, you must make $q - \frac{1}{4}pp = \alpha$, $r - \frac{1}{2}p\alpha = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$, $v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \epsilon$, $a - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma = \zeta$, $b - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma = \eta$, $c - \frac{1}{2}\gamma\delta = \theta$, $d - \frac{1}{4}\delta\delta = \kappa$. And seek such a common Divisor to the five Terms, 2ϵ, 2ζ, 8η, 4θ, 8κ, as is an Integer, and not a Square, but which shall leave 1 when divided by 4, if any one of the Terms p, r, t, a, c be odd. See *Newton*'s Alg. *p.* 218.

Problem XXIV.

519. To resolve a Cubic Equation, where the second Term is wanting.

Effection.

Let there be propos'd the Cubic Equation $x^3 * + qx + r = 0$; the second Term whereof is wanting: For that every Cubic Equation may be reduc'd to this Form, is evident from what we have said above. Let x be suppos'd $= a + b$. Then will $a^3 + 3aab + 3abb + b^3$ (that is x^3) $= qx + r = 0$. Let $3aab + 3abb$ (that is, $3abx$) $+ qx = 0$, and then will $a^3 + b^3 + r = 0$. By the former

mer Equation b is $=-\frac{q}{3a}$, and cubically $b^3=-\frac{q^3}{27a^3}$. Therefore by the latter, $a^3-\frac{q^3}{27a^3}+r=0$, or $a^6+ra^3=\frac{q^3}{27}$, and by the Extraction of the adfected Quadratic Root, $a^3=-\frac{1}{2}r\pm\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}$. Extract the Cubic Root and you'll have a. And above, you had $\frac{-q}{3a}=b$, and $a+b=x$. Therefore $a-\frac{q}{3a}$ is the Root of the Equation propos'd.

For Example, let there be propos'd the Equation $y^3-6yy+6y+12=0$. To take away the second Term of this Equation, make $x+2=y$, and there will arise $x^{3*}-6x+8=0$. Where q is $=-6$, $r=8$, $\frac{1}{4}rr=16$, $\frac{q^3}{27}=-8$, $a^3=-4\pm8^{\frac{1}{2}}$, $a-\frac{q}{3a}=x$, and $x+2=y$, that is, $2+\overline{-4\pm8^{\frac{1}{2}}}^{\frac{1}{3}}+\frac{2}{\overline{-4\pm8^{\frac{1}{2}}}^{\frac{1}{3}}}=y$

Scholium XVI.

520. And after this Way the Roots of all Cubical Equations may be extracted wherein q is Affirmative; or also wherein q is Negative, and $\frac{q^3}{27}$ not greater than $\frac{1}{4}rr$, that is, where two of the Roots of the Equation are impossible. But where q is Negative, and $\frac{q^3}{27}$ at the same time greater than $\frac{1}{4}rr$, $\overline{\frac{1}{4}rr-\frac{q^3}{27}}^{\frac{1}{2}}$ becomes an impossible Quantity, and so the Root of the Equation x or y will in this Case be impossible, *viz.* in this Case there are three possible Roots, which all of them are alike with respect to the Terms of the Equation q and r, and are indifferently denoted by the Letter x and y, and consequently all them may be extracted by the same Method, and express'd the same Way as any one is extracted or expressed; but it is impossible to express all three by the Law aforesaid. The Quantity $a-\frac{q}{3a}$ whereby x is denoted, cannot be manyfold, and for that Reason the Supposition

position that x, in this Case where it is triple, may be equal to the Binomial $a-\frac{q}{3a}$, or $a+b$, the Cubes of whose Terms a^3+b^3 are together $=r$, and the triple Rectangle $3ab$ is $=q$, is plainly impossible; and it is no Wonder that from an impossible Hypothesis, an impossible Conclusion should follow.

Scholium XVII.

521. There is, moreover, another Way of expressing these Roots, *viz.* from a^3+b^3+r, that is, from nothing take

a^3+r, or $\frac{1}{2}r\pm\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}$, and there will remain $b^3=$

$-\frac{1}{2}r\mp\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}$. Therefore

$$a=\overline{-\frac{1}{2}r+\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}}^{\frac{1}{3}}$$

$$b=\overline{-\frac{1}{2}r-\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}}^{\frac{1}{3}}.$$

$$a=\overline{-\frac{1}{2}r-\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}}^{\frac{1}{3}}$$

$b=\overline{-\frac{1}{2}r+\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}}^{\frac{1}{3}}$ and consequently the Sum

of these $\overline{-\frac{1}{2}r+\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}}^{\frac{1}{3}}+$

$\overline{-\frac{1}{2}r-\overline{\frac{1}{4}rr+\frac{q^3}{27}}^{\frac{1}{2}}}^{\frac{1}{3}}$ will be x.

Problem XXV.

522. To reduce a Biquadratic Equation $x^4+qx^2+rx+s=0$, wanting the second Term to a Cubic one.

 Effection.

Effection.

Suppose this Equation to be generated by the Multiplication of these two $xx+ex+f=0$, and $xx-ex+g=0$, that is, to be the same with this $x^4 * \begin{smallmatrix}+f\\+g\\-ee\end{smallmatrix} xx \begin{smallmatrix}+eg\\-ef\end{smallmatrix} x+fg=0$, and comparing the Terms you'll have $f+g-ee=q$, $eg-ef=r$, and $fg=s$. Wherefore $q+ee=f+g$, $\frac{r}{e}=g-f$, $\frac{q+ee+\frac{r}{e}}{2}=g$, $\frac{q+ee-\frac{r}{e}}{2}=f$,

$$\frac{qq+2eeq+e^4-\frac{rr}{ee}}{4}\,(=fg)=s,$$

and by the Reduction $e^6+2qe^4 \begin{smallmatrix}+qq\\-4s\end{smallmatrix} ee-rr=0$.

For ee write y, and you'll have $y^3+2qyy \begin{smallmatrix}+qq\\-4s\end{smallmatrix} y-rr=0$, a Cubic Equation, whose second Term may be taken away, and then the Root extracted either by the precedent Rule or otherwise. Then that Root being had, you must go back again, by putting $y^{\frac{1}{2}}=e$, $\frac{q+ee-\frac{r}{e}}{2}=f$, $\frac{q+ee+\frac{r}{e}}{2}=g$, and the two Equations $xx+ex+f=0$; and $xx-ex+g=0$, their Roots being extracted, will give the four Roots of the Biquadratic Equation $x^4+qxx+rx+s=0$, *viz.* $x=-\frac{1}{2}e\pm\overline{\tfrac{1}{4}ee-f}\,^{\frac{1}{2}}$, and $x=\frac{1}{2}e\pm\overline{\tfrac{1}{4}ee-g}\,^{\frac{1}{2}}$. Where note, that if the four Roots of the Biquadratic Equation are possible, the three Roots of the Cubic Equation $y^3+2qyy \begin{smallmatrix}+qq\\-4s\end{smallmatrix} y-rr=0$ will be possible also, and consequently cannot be extracted by the precedent Rule. And thus, if the affected Roots of an Equation of five or more Dimensions are converted into Roots that are not affected, the middle Terms of the Equation being taken away, that Expression of the Roots will be always impossible, where more than one Root in an Equation of odd Dimensions are possible, or more than two in an Equation of even Dimensions, which cannot be reduc'd by the Extraction of the Surd Quadratic Root, by the Method laid down above.

Scholium XVIII.

523. Monsieur *Des Cartes* taught how to reduce a Biquadratic Equation by the Rules last deliver'd. *E. g.* Let there be propos'd the Equation reduc'd above, $x^4-x^3-5xx+12x-6=0$. Take away the second Term by writing

$y+$

$v+\frac{1}{4}$ for x, and there will arise $v^4-\frac{43}{8}vv+\frac{75}{8}v-\frac{851}{256}=0$. To take away the Fractions, write $\frac{1}{4}z$ for v, and there will arise $z^4-86zz+600z-851=0$. Here $-86=q$, $600=r$, and $-851=s$, and consequently $y^3+2qyy\,\begin{matrix}+qq\\-4s\end{matrix}\,y-rr=0$, and substituting what is equivalent, you'll have $y^3-172yy+10800y-360000=0$. Where trying all the Divisors of the last Term $1, -1, 2, -2, 3, -3, 4, -4, 5, -5$, and so onwards to 100, you'll find at length $y=100$. Which yet may be found far more expeditiously by our Method above deliver'd. Then having got y, its Root 10 will be e, and $\frac{q+ee-\frac{r}{e}}{2}$, that is $\frac{-86+100-60}{2}$ or -23, will be f, and $\frac{q+ee+\frac{r}{e}}{2}$, or 37 will be g, and consequently the Equations $xx+ex+f=0$, and $xx-ex+g=0$, writing z for x, and substituting equivalent Quantities, will become $zz+10z-23=0$, and $zz-10z+37=0$. Restore v in the room of $\frac{1}{4}z$, and there will arise $vv+2\frac{1}{2}v-\frac{23}{16}=0$, and $vv-2\frac{1}{2}v+\frac{37}{16}=0$. Restore, moreover, $x-\frac{1}{4}$ for v and there will come out $xx+2x-2=0$, and $xx-3x+3=0$, two Equations; the four Roots whereof $x=-1\pm 3^{\frac{1}{2}}$, and $x=1\frac{1}{2}\mp\left(-\frac{3}{4}\right)^{\frac{1}{2}}$, are the same with the four Roots of the Biquadratic Equation propos'd at the Beginning, $x^4-x^3-5xx+12x-6=0$. But these might have been more easily found by the Method of finding Divisors, explain'd before.

CHAP. X.

Of Mixed Equations.

DEFINITION XXVII.

524. A *Mixed Equation* is that which contains in it more unknown Quantities than one: as the Equation $2aa+ae=b-a$, or $\frac{a}{e}+y=b$.

DEFINITION XXVIII.

525. The Extermination of an unknown Quantity out of an Equation, is the bringing it to one Side in two or more given Equations, so that one or more new Equations may be had without that unknown Quantity.

COROLLARY

Corollary.

526. Whence it follows, that there must be at least as many given Equations as unknown Quantities, before the Value of all the unknown Quantities can be found or exterminated.

Problem XXVI.

527. To exterminate an unknown Quantity by an Equality of its Values.

Example 1.

From the	1	$a+1=e+y$	
Equations	2	$a-e=\frac{y}{2}$	find a, e, y.
	3	$a+y=2e$	
$1-e$	4	$a+1-e=y$	From whence we have two new Equations with y exterminated.
2×2	5	$2a-2e=y$	
$3-a$	6	$2e-a=y$	
$4,\ 5$	7	$a+1-e=2a-2e$	(In. 21.) the first Equation with y exterminated.
$7+2e$	8	$a+1+e=2a$	
$8-a+1$	9	$e=a-1$	
$5,\ 6$	10	$2e-a=2a-2e$	(In. 21.) the second Equation with y exterminated.
$10+2e$	11	$4e=3a$	
$11\div 4$	12	$e=\frac{3a}{4}$	
$9,\ 12$	13	$a-1=\frac{3a}{4}$	(In. 21.) the last Equation with y and e exterminated from the 9th and 12th Steps.
13×4	14	$4a-4=3a$	
$14+4$	15	$4a=3a+4$	
$15-3a$	16	$a=4$	
9.	17	$e=a-1=3$	
4.	18	$y=a+1-e=2$	

Example 2.

From the given Equations.	1	$\frac{cca-bcy}{b}=bc+ay$ to find a and y
	2	$ca+\frac{byy}{d}=2bb$
$1\times b$	3	$cca-bcy=bbc+bay$
$3+bcy-bay$	4	$cca-bay=bbc+bcy$
$4\div\overline{cc-by}$	5	$a=\frac{bbc+bcy}{cc-by}$
$2\times d$	6	$dca+byy=2dbb$
$6-byy$	7	$dca=2dbb-byy$
$7\div dc$	8	$a=\frac{2dbb-byy}{dc}$
5, 8	9	$\frac{bbc+bcy}{cc-by}=\frac{2dbb-byy}{dc}$ an Equation with a exterminated.
$9\times dc\times\overline{cc-by}$	10	$b^2c^2d+bc^2dy=2b^2c^2d-bc^2y^2-2b^3dy+b^2y^3$
or	11	$b^2y^3-bc^2y^2-2b^3dy-bc^2dy+b^2c^2d=0$
$11\div b^2$	12	$y^3-\frac{c^2}{b}y^2-\frac{2b^2d+c^2d}{b}y+ccd=0$, a Cubic Equation whose

Root y may be found as is taught (In. 554 or 556.) and thence from Step the 5th or 8th will be had the Value of a.

Problem XXVII.

528. To exterminate an unknown Quantity by substituting its Value for it.

Example 1.

From	1	$ae^2=bb-c^2$ to find a and e
	2	$aa-ee=bc$
$1\div a$	3	$e^2=\frac{bb-cc}{a}$ which substitute for ee Step 2.
Sub. $\frac{bb-cc}{a}$ Step 2.	4	$aa-\frac{bb-cc}{a}=bc$
$4\times a$	5	$a^3-bb+cc=bca$
$5+bb-cc$	6	$a^3-bca=bb-cc$ which resolved will give the Value of a.
3 ɯ 2	7	$e=\overline{\frac{bb-cc}{a}}\Big\vert^{\frac{1}{2}}$

Example

Example 2.

From	1	$baa+bba=e^3$ } to find a and e
	2	$ae-ba=be$
$2\div\overline{e-b}$	3	$a=\frac{be}{e-b}$ which substitute for a and a^2, Step the first.
3 ☉^{2}	4	$a^2=\frac{bbee}{ee-2be+bb}$
Sub. $\frac{be}{e-b}$ Step 1.	5	$\frac{b^3e^2}{e^2-2be+b^2}+\frac{b^3e}{e-b}=e^3$
$5\times\overline{e^2-2be+b^2}$	6	$b^3e^2+b^3e^2-b^4e=e^5-2be^4+b^2e^3$
$6\div e$	7	$2b^3e-b^4=e^4-2be^3+b^2e^2$
or	8	$e^4-2be^3+b^2e^2-2b^3e+b^4=0$

Example 3.

From	1	$ba=\frac{cca+c^3}{e}$ } to find a and e
	2	$a=\frac{be}{a+c}$
1×2	3	$ba^2=\frac{bc^2ae+bc^3e}{ae+ce}=bc^2$
$3\div b$	4	$a^2=c^2$
4 ш^2	5	$a=c$
Sub. $c=a$ Step 2.	6	$c=\frac{be}{2c}$
$6\div\frac{b}{2c}$	7	$\frac{2cc}{b}=e$

PROBLEM XXIX.

529\. To exterminate an unknown Quantity of several Dimensions in each given Equation.

From	1	$a^2+6ee=12a$ } to find a and e.
	2	$a^2+6ae=18$
$1-6e^2$	3	$a^2=12a-6e^2$
$2-6ae$	4	$a^2=18-6ae$
3, 4,	5	$12a-6ee=18-6ae$
$5+6ae+6ee$	6	$12a+6ae=18+6ee$
$6\div 6$	7	$2a+ae=3+ee$
$7\div\overline{2+e}$	8	$a=\frac{3+ee}{2+e}$ } to be substituted in Step 1.
8 ☉^{2}	9	$a^2=\frac{9+6ee+e^4}{4+4e+e^2}$
whence	10	$\frac{9+6ee+e^4}{4+4e+e^2}+6ee=\frac{36+12ee}{2+e}$
$10\times\overline{4+4e+e^2}$	11	$9+6ee+e^4+24e^2+24e^3+6e^4=72+24ee+36e+12e^3$
	12	$7e^4+12e^3+6e^2-36e-63=0$. Whence e may be found by (In. 509.)

Example

Example 2.

From	1	$e^2=\frac{2a^2e}{b}+a^2$ } find a and e.
	2	$e^2=2ae+\frac{a^4}{bb}$
$1-\frac{2a^2}{b}e$	3	$ee-\frac{2aa}{b}e=a^2$
$3+\frac{a^4}{b^2}$	4	$ee-\frac{2aa}{b}e+\frac{a^4}{b^2}=a^2+\frac{a^4}{b^2}$
$4\,w^2$	5	$e-\frac{a^2}{b}=\left(a^2+\frac{a^4}{b^2}\right)^{\frac{1}{2}}$ (In. 506.)
$5+\frac{a^2}{b}$	6	$e=\left(a^2+\frac{a^4}{b^2}\right)^{\frac{1}{2}}+\frac{a^2}{b}$
$2-2ae$	7	$e^2-2ae=\frac{a}{b^2}$
$7+a^2$	8	$e^2-2ae+a^2=a^2+\frac{b^4}{b^2}$
$8\,w^2$	9	$e-a=\left(a^2+\frac{aaaa}{bb}\right)^{\frac{1}{2}}$ (In. 506.)
$9+a$	10	$e=\left(a^2+\frac{aaaa}{b^2}\right)^{\frac{1}{2}}+a$
6, 10,	11	$\left(a^2+\frac{a^4}{bb}\right)^{\frac{1}{2}}+\frac{a^2}{b}=\left(a^2+\frac{a^4}{b^2}\right)^{\frac{1}{2}}+a$
$11-\left(a^2+\frac{a^4}{b^2}\right)^{\frac{1}{2}}$	12	$\frac{a^2}{b}=a$
$12\times b$	13	$a^2=ba$
$13\div a$	14	$a=b$ &c.

Example 3.

From	1	$a+e+\frac{ee}{a}=20$ } find a and e
	2	$a^2+e^2+\frac{e^4}{a^2}=140.$
$1-e$	3	$a+\frac{ee}{a}=20-e$

$3 ⊙^2$	4	$a^2+2e^2+\frac{e^4}{a^2}=400-40e+ee$
$4-ee$	5	$a^2+e^2+\frac{e^4}{a^2}=400-40e$
$2,\ 5$	6	$140=400-40e$
$6\div\overline{10}$	7	$14=40-4e$
$\therefore$	8	$4e=40-14=26$
$8\div\overline{4}$	9	$e=\frac{26}{4}=6\frac{1}{2}$ to be substituted in the first Step, &c.

Example 4.

From	1	$a^3-a^2e=3e$ } find a and e
	2	$a^2+ae+3=ee$
$2-ae$	3	$e^2-ae=a^2+3$
$3+\frac{aa}{4}$	4	$ee-ae+\frac{aa}{4}=aa+\frac{aa}{4}+3=\frac{5aa}{4}+3$
$4 ω^2$	5	$e-\frac{1}{2}a=\overline{\frac{5a^2}{4}+3}^{\frac{1}{2}}$
$5+a\frac{1}{2}$	6	$e=\overline{\frac{5aa}{4}+3}^{\frac{1}{2}}+\frac{1}{2}a$
Sub. the Value of e into the first Step.	7	$a^3-a^2\ \overline{\frac{5aa}{4}+3}^{\frac{1}{2}}+\frac{1}{2}a^3=3\ \overline{\frac{5aa}{4}+3}^{\frac{1}{2}}+1\frac{1}{2}a$
$7\times\overline{2}$	8	$3a^3-2a^2\ \overline{\frac{5aa}{4}+3}^{\frac{1}{2}}=6\ \overline{\frac{5aa}{4}+3}^{\frac{1}{2}}+3a$
$\therefore$	9	$3a^3-3a=\overline{6+2a^2}\times\overline{\frac{5aa}{4}+3}^{\frac{1}{2}}$
$9\div\overline{6+2a^2}$	10	$\frac{3a^3-3a}{6+2a^2}=\overline{\frac{5aa}{4}+3}^{\frac{1}{2}}$
$10 ⊙^2$	11	$\frac{9a^6+18a^4+9a^2}{4a^4+24a^2+36}=\frac{5aa}{4}+3$
$11\times\overline{4}$	12	$\frac{9a^6-18a^4+9a^2}{a^4+6a^2+9}=5a^2+12$ (In. 115.)
$12\times\overline{a^4+6a^2+9}$	13	$9a^6-18a^4+9a^2=5a^6+42a^4+117a^2+108$
$13-5a^6=42a^4-117a^2$	14	$4a^6-60a^4-108a^2=108$
$14\div4$	15	$a^6-15a^4-27a^2=27$
Substitute	16	$a^2=y$
Then	17	$y^3-15y^2-27y=27$
$16 ω^2$	18	$a=y^{\frac{1}{2}}$

Example

Example 5.

From	1	$2a^4-5a^3e+3a^2e^2-2ae^3+e^4-37=0$ to find a and e
	2	$2a^2e^2-ae^3-12e-15=0$
$2\div 2$	3	$a^2e^2-\frac{1}{2}ae^3=6e+7.5$
$3\div ee$	4	$a^2-\frac{1}{2}ae=\frac{6e+7.5}{ee}$
$4+\frac{1}{16}ee$	5	$a^2-\frac{1}{2}ae+\frac{1}{16}ee=\frac{eeee+96e+120}{16ee}$
$5\,\omega^2$	6	$a-\frac{1}{4}e=\overline{\frac{e^4+96e+120}{16ee}}^{\frac{1}{2}}$
$6+\frac{e}{4}$	7	$a=\frac{\overline{e^4+96e+120}^{\frac{1}{2}}+e^2}{4e}$
Make	8	$e^4+96e+120=y^2$, or $\overline{e^4+96e+120}^{\frac{1}{2}}=y$
then	9	$a=\frac{y+ee}{4e}=\frac{64eeey+64e^5}{256e^4}$
$9\odot^2$	10	$a^2=\frac{16e^2y^2+32e^4y+16e^6}{256e^4}$
$9\odot^3$	11	$a^3=\frac{4ey^3+12e^3y^2+12e^5y+4e^4}{256e^4}$
$9\odot^4$	12	$a^4=\frac{y^4+4y^3e^2+6y^2e^4+4ye^6+e^8}{256e^4}$
Sub. for a in the 1st Step dividing by $256eeee$	13	$2y^4-12y^3e^2-84ye^6+158e^8-9472e^4=0$
$13\div 2$	14	$y^4-6y^3e^2-42ye^6+79e^8-4736e_4=0$
$14+6y^3e^2+42ye^6$	15	$y^4+79e^8-4736e^4=6y^3e^2+42ye_6$
$15\div 6y^2e^2$ &c.	16	$\frac{y^4+79e^8-4737e^4}{6yyee+42e^6}=y$
Sub. for y	17	$\frac{80e^8+192e^5-4496e^4+9216e^2+23040e+14400}{48e^6+576e^3+720e^2}=y$
$17\div\frac{16}{16}$	18	$\frac{5e^8+12e^5-281e^4+576e^2+1440e+900}{3e^6+36e^3+45e^2}=y$
$18\odot^2$	19	$\overline{25e^{16}+120e^{13}-2810e^{12}+5904e^{10}+7656e^9+87961e^8+138248^7}$ $\overline{-289152e^6-787680e^5-174024e^4+1658880e^3+3110400e^2+}$ $\overline{2592000e+810000}\div\overline{9e^{12}+216e^9+270e^8+1296e^6+3240e^5}$ $\overline{+2025e^4}=yy=e+96e^4+120$

$\overline{19\times 9e^{12}+\&c.}$	20	$25e^{16}+120e^{13}-2810e^{12}+5904e^{10}+7656e^{9}+87961e^{8}+13824e^{7}-289152e^{6}-787680e^{5}-174024e^{4}+1658880e^{3}+3110400e^{2}+2592000e+810000=9e^{16}+1080e^{13}+1350e^{12}+22032e^{10}+55080e^{9}+34425e^{8}+124416e^{7}+466560e^{6}+583200e^{5}+243000e^{4}.$
Whence	21	$16e^{16}-960e^{13}-4160e^{12}-16128e^{10}-47424e^{9}+53536e^{8}-110592e^{7}-755712e^{6}-1370880e^{5}-417024e^{4}+1658880e^{3}+3110400e^{2}+2592000e+810000=0.$
$21\div\overline{16}$	22	$e^{16}-60e^{13}-260e^{12}-1008e^{10}-2964e^{9}+3346e^{8}-6912e^{7}-47232e^{6}-85680e^{5}-26064e^{4}+103680e^{3}+194400e^{2}+162000e+50625=0.$

An Equation of ſixteen Dimenſions, whoſe Root e will be found to be 5 (In. 509.) whence $a=\frac{\overline{e^4+96e+120^2}^{\frac{1}{2}}+e^2}{4e}=3$ (Step. 7th.)

Scholium XIX.

530. Hither may be referred Sir *Iſaac Newton*'s Rules for the Extermination of unknown Quantities in adfected Equations. *Vid. Newton*'s Algebra, p. 65, or in the beſt *Latin* Edition, p. 73.

CHAP. XI.

Of bringing Queſtions into Equations.

Partition VII.

531. *Algebraical Queſtions* are of two Sorts, *Determinate* and *Indeterminate*, according to the Number of Equations and unknown Terms whereof they conſiſt.

Definition XXIX.

532. A *Determinate Queſtion* is that which conſiſts of as many independent Equations, as unknown Quantities: As if it were required to find what two Numbers thoſe are, a and e. whereof $ae=24$ and $\frac{a}{e}=6$. Or to find what three Numbers thoſe are, whereof $a+e=25$, $a+y=28$, $e+y=31$. Which kind of Queſtions are called *Determinate*, becauſe they admit of no more Anſwers than the Number of the Dimenſions of the laſt Quantity ſought.

Definition

Definition XXX.

533. An *Indeterminate Question* is that which consists of more unknown Quantities than Equations: As if it were required to find what two Numbers those are, a and e, whereof $ae=24$: Or to find what three Numbers those are, whereof $a+e=25$, and $a+y=28$: Which Sort of Questions are called *Indeterminate*, because they admit each of innumerable Answers. *Ex. gr.* Any two Numbers whose Product is 24 will answer the former, and in answering the latter, any Number less than 25 may be assumed for one of the unknown Quantities.

Scholium XX.

534. If in any Question the Number of Equations be greater than the Number of unknown Quantities; it is odds but some of the Equations are contradictory, and consequently the Solution of such Question is impossible.

Partition VIII.

535. Again, Algebraical Questions may be divided into *General* and *Particular*.

Definition XXXI.

536. A *General Question* is one expressed in general Terms, the Solution of which affords Theorems for particular Cases; as when it is required to investigate a Theorem for Determining any three Numbers or Quantities a, e, y from the Sums of every two of them given, *viz.* $a+e=b$, $a+y=c$, $e+y=d$, where for b, c, and d, may be assumed any three Numbers at Pleasure.

Definition XXXII.

537. A *Particular Question* is expressed in particular Terms; as when it is required to find three Numbers a, e, y, on these Conditions that $a+e=25$, $a+e=28$, and $e+y=31$.

Problem XXX.

538. To bring a Question to an Equation.

Effection.

The whole Art of bringing Questions into Equations consists in a due Expression of all their Quantities by proper Species, (the known Quantities by Consonants (or Numbers) and the unknown ones by Vowels (In. 530.) which is to be learned by Example rather than Precept.

Example 1. A Lady seeing divers poor Persons at her Door was willing to distribute some Money among them, but when she Number'd them, she found that she wanted Six-pence to give Four-pence a piece to them; she therefore gave to each Three-pence, and had Two-pence remaining: What was the Number of poor People, and what Money had she in her Pocket?

A Lay

The Questions in Words.	*The same in Species.*
A Lady seeing divers poor Persons at her Door for whose Numbers put	a
was willing to distribute some Money among them, for the Quantity of which put	e
But when she Number'd them she wanted Six-pence to give them Four-pence a-piece, *i. e.*	$4a=e+6.$
She therefore gave to each Three-pence, and had Two-pence remaining, *i. e.*	$3a=e-2$

The Question then consists of two Equations, and two unknown Quantities a, e. Whence by (In 527.) a will be found $=8$ Pence and $e=26$ Pence or 2 *s.* 2*d.*

Example 2. One bought a Horse, and Sold it again for eleven Pounds, in the selling it gained as much *per. Cent.* as the Horse cost him. What did he give for the Horse?

The Questions in Words.	*The same in Species.*
One bought a Horse whose Price was	a
And sold it again for 11 *l.* or	b
So that his Gain was	$b-a$
In the selling of which he gain'd as much by the 100 *l.* or	c
As the Horse cost him, *i. e.*	$a{:}b-a=c{:}a$

Whence we have this Equation $aa=bc-ac$ (In. 189.) which by due Reductions gives $a=\frac{\overline{4bc+cc}^{\frac{1}{2}}-c}{2}=10\,l.$ (In. 506.)

Example 3. A Father and his Son went to the Wood for each a Burthen of Sticks, and in their return Home the Son complained that he was over-loaded; to whom the Father replied, if I take ten of your Sticks, then shall I have twice the Number of Sticks that you have, and if you take ten of mine we shall have an equal Number. It is required to find the Number of Sticks that each had.

Here if a be put to represent the Number of Sticks which the Father had, and e the Number of Sticks which the Son had; and if for 10 be put b, then will the Question be expressed by these two Equations:

$$\begin{array}{c|l} 1 & a+b=2\times\overline{e-b}=2e-2b \\ 2 & a-b=e+b \end{array}$$

Whence by due Reduction, it will be found that $e=5b=50$ and $a=e+2b=7b=70$. (In. 527.)

Example

Example 4. A Man had three Horses whose Values were as follows. If eleven Guineas were added to the Value of the worst, it would equal the Value of both the others; if eleven Guineas were added to the second, it would equal twice the Value of the two others; and if eleven Guineas were added to the Value of the best, it would equal thrice the Value of the other two. What was the Value of each?

Put a for the Value of the worst, e for the Value of the second, and y for the Value of the best, then make $b=11$, and the Question will be expressed by three Equations as follows.

$$\begin{array}{c|l} 1 & a+b=e+y \\ 2 & e+b=2a+2y \\ 3 & y+b=3a+3y \end{array}$$

Therefore by due Reduction $y=\frac{7b}{11}=7$ Guineas, $e=3b-4y=\frac{5b}{11}=5$ Guineas, and $a=e+y-b=\frac{b}{11}=1$ Guinea.

Example 5. A Shepherd being asked the Price of his hundred Sheep, made Answer; I have not a hundred, but if I had as many, half as many, two and a half, then shou'd I have just a hundred. What was the Number of his Sheep?

Here putting a for the Number of Sheep the Question is expressed by this one Equation $a+a+\frac{a}{2}+2\frac{1}{2}=100$. Whence $a=39$.

Example 6. One being asked what a-Clock it was, made Answer, that the Time then past from Noon was equal to two fifths of the Time remaining to Midnight. What was the Hour after Noon?

Put $b=12$, and let a represent the Hours from Noon, and consequently $b-a$ the Time to Midnight. Then

$$\begin{array}{c|l} 1 & a=\frac{2b-2a}{5} \text{ by the Question.} \\ 2 & a=\frac{2b}{7}=3\frac{3}{7} \text{ Hours after Noon.} \end{array}$$

Whence

Example 7. 'Tis required to divide $b=100$ twice into two Parts, so that the Major Part of the first Division is double the Minor Part of the second Division; and the Major Part of the second Division wanting one, is to the Minor Part of the first Division as 9 to 4.

Put $a=$ the Major Part of the first Division, then will $b-a$ equal the Minor Part of the same Division; and by the Question the Minor Part of the second Division will be $\frac{1}{2}a$, and consequently the Major Part of the same Division $b-\frac{1}{2}a$

Then	1	$b-\frac{1}{2}a-1 : b-a=9 : 4$ by the Question.
Or (by In)	2	$4b-2a-4=9b-9a$
Whence	3	$a=\frac{5b+4}{7}=72$ the Major } Part of the first Division.
∴	4	$b-a=28$ the Minor } Part of the first Division.
	5	$\frac{a}{2}=36$ the Minor } Part of the second Division.
	6	$b-\frac{a}{2}=64$ the Major } Part of the second Division.

Example 8. A Gentleman hired a Servant for a Year for 120 Shillings together with a Riding Coat at a certain Price; but when seven Months were expired, the Master falling at variance with the Servant puts him away, and gives him the Riding Coat with 50 Shillings, which was full Satisfaction for the Time. What was the Value of the Riding Coat?

Put $b=120$, $c=7$, and $d=50$, and for the Price of the Riding Coat put a

Then	1	$b+a=$ his Wages for the Year
$1\div 12$	2	$\frac{b+a}{12}=$ his Wages for a Month
$2\times c$	3	$\frac{cb+ca}{12}=$ his Wages for seven Months
∴	4	$\frac{cb+ca}{12}=a+d$ by the Question
Whence	5	$a=\frac{cb-12d}{12-c}=48$ Shillings, the Price of the Coat.

Example 9. A General having set his Soldiers in a Square Battalion had $=500$ Soldiers to spare; but to encrease the Square, so that its Side might bonsist of $c=1$ Soldier more than it did before, he wanted $d=29$ Soldiers. How many Soldiers were in the whole Army.

For the Number of Soldiers in the Side of the first Square put a

Then	1	$a+c=$ the Soldiers in the Side of the second Square
	2	$aa+b=$ the whole Army
	3	$aa+b+d=a^2+2ca+c^2$ by the Question
$3-aa$	4	$b+d=2ca+cc$
Whence	5	$a=\frac{b+d-cc}{2c}=264.$
∴	6	$aa+b=70396$, the Number of Soldiers in the Army.

Example 10. A Man playing at Hazard won the first Throw, just so much Money as he had in his Pocket, the second Throw he won the Square Root of what he then had, and $b=5s$. more; the third Throw he won the Square of all he then had; after which his whole Sum was $c=2256s$. It is required to find

find what Money he had in his Pocket, when he began to play. *Ward's* Introduction, p. 225.

Make	1	$2aa=$ the first Sum he had in his Pocket
1×2	2	$4aa=$ the Sum after the first Throw
∴	3	$2a+5=$ the winning at the second Throw
2×3	4	$4aa+2a+5=$ the Sum after the second Throw.
Subst.	5	$4aa+2a+5=e$
$5 \odot 2$	6	$ee=$ the winning of the third Throw
$5+6$	7	$ee+e=2256$ Shillings by the Question
$7\times\frac{1}{4}$	8	$ee+e+0.25=2256.25$ (In. 506.)
$8\,\omega\,2$	9	$e+0.5=2256.25^{\frac{1}{2}}=47.5$
$9-\frac{1}{2}$	10	$e=47$
5, 10,	11	$4aa+2a+5=47$
$11-5$	12	$4aa+2a=42$
$12\div 4$	13	$aa+\frac{1}{2}a=10.\frac{1}{2}$
$13+\frac{1}{16}$	14	$aa+\frac{1}{2}a+\frac{1}{16}=10\frac{1}{2}+\frac{1}{16}=\frac{169}{16}$ (In. 506.)
$14\,\omega\,2$	15	$a+\frac{1}{4}=\frac{13}{4}$
$15-\frac{1}{4}$	16	$a=\frac{13-1}{4}=\frac{12}{4}=3$
$16 \odot 2$	17	$aa=9$ Shillings
17×2	18	$2aa=18$ Shillings, the Sum he had first in his Pocket.

Scholium XXI.

539. From the last Example belonging to the foregoing Problem it appears of how much Consequence a convenient Hypothesis is for answering any Question: For if the first Sum had been represented by a or aa, the Equation, which expresses the Question, wou'd have been encumber'd with Surds; because the Sum after the first Throw could not have its Root expressed in rational Terms, as the Question requires; but this is remedied by putting $2aa$ for that Sum as above.

Definition XXXIII.

540. The Equations by which Indeterminate Questions or Problems are expressed, may be distinguished by the Name of *Equalities*; those which are expressed by one Equation by the Name of *single Equalities*; those which are expressed by two Equations by *double Equalities*; by three Equations, *Triple Equalities*, &c.

Partition IX.

541. And those again may be divided into *Homologous*, and *Heterologous*.

Definition

DEFINITION XXXIV.

542. By *Homologous Equalities* are meant ſuch as have all their unknown Quantities of the ſame Dimenſions; as $ba+ce=da-de$, $aa-ee=d$, $a^3+e^3=x$, *&c.*

DEFINITION XXXV.

543. By *Heterologous Equalities* are meant ſuch as have their unknown Quantities of different Dimenſions: as $a^2-e=x$, $a^2+e^2=e^3$, *&c*

PARTITION X.

544. Laſtly, *Homologous Equalities* may be divided into *Lateral*, *Quadratic*, *Cubic*, &c. according as their unknown Quantities are of one, two, three, *&c.* Dimenſions.

The End of the fourth Part.

ARITHME-

ARITHMETICAL INSTITUTIONS.

PART V.

The APPLICATION of SPECIES ALGORISM to the INVESTIGATION of THEOREMS.

CHAP. I.

Containing certain promiscuous PROBLEMS.

PROBLEM I.

545\. TO raise *Theorems* for finding any two Quantities a, e, from their Sum s and Difference d given.

Effection.

	1	$a + e = s$ } *By the Question.*
	2	$a - e = d$
$1 + 2$	3	$2a = s + d$
$3 \div 2$	4	$a = \frac{s+d}{2}$ *The* Theorem *for the greater Number.*
$1 - 2$	5	$2e = s - d$
$5 \div 2$	6	$e = \frac{s-d}{2}$ *The* Theorem *for the lesser Number.*

Ex. gr. If $s = 35$, and $d = 21$, then $a = \frac{s+d}{2} = 28$, and $e = \frac{s-d}{2} = 7$.
Proof, $19 + 12 = 31 = s$, and $19 - 12 = 7 = d$.

Corollary I.

546. If $2s$ be put to represent the Sum of any two Numbers, and $2d$ for the lesser, then will $s+d$ give the greater Number sought, and $s-d$ the lesser.

Problem II.

547. To raise *Theorems* for finding any two Numbers a, e, from their Sum s, and Product p given.

Effection.

	1	$a + e = s$ } *By the Question.*
	2	$ae = p$ }
$1 \odot^2$	3	$a^2 + 2ae + ee = ss$
$2 + 4$	4	$4ae = 4p$
$3 - 4$	5	$a^2 - 2ae + ee = ss - 4p$
$5\ w\ ^2$	6	$a - e = \overline{ss - 4p}^{\frac{1}{2}}$
$1 + 6$	7	$2a = s + \overline{ss - 4p}^{\frac{1}{2}}$
$7 \div 2$	8	$a = \frac{s + \overline{ss - 4p}^{\frac{1}{2}}}{2}$ *The* Theorem *for the greater Number.*
$1 - 6$	9	$2e = s - \overline{ss - 4p}^{\frac{1}{2}}$
$9 \div 2$	10	$e = \frac{s - \overline{ss - 4p}^{\frac{1}{2}}}{2}$ *The* Theorem *for the lesser.*

Ex. gr. If $s = 35$, and $p = 196$, then $a = \frac{s + \overline{ss - 4p}^{\frac{1}{2}}}{2} = 28$, $e = \frac{s - \overline{ss - 4p}^{\frac{1}{2}}}{2} = 7$.

Problem. III.

548. To raise *Theorems* for finding any two Numbers a, e, from their Sum s and Quotient q given.

Effec-

Effection.

	1	$a + e = s$	*By the Question.*
	2	$\frac{a}{e} = q$	
$2 \times e$	3	$a = qe$	
$1 - 3$	4	$e = s - qe$	
$4 + qe$	5	$qe + e = s$	
$5 \div \overline{q+1}$	6	$e = \frac{s}{q+1}$	*The* Theorem *for finding the lesser Number.*
2×6	7	$a = \frac{qs}{q+1}$	*The* Theorem *for the greater Number.*

Ex. gr. If $s = 35$, $q = 4$, then $a = \frac{qs}{q+1} = 28$, $e = \frac{s}{q+1} = 7$.

PROBLEM IV.

549. To raise *Theorems* for finding any two Numbers a, e, from their Sum s, and Sum of their Squares z given.

Effection.

	1	$a + e = s$	*By the Question.*
	2	$a^2 + e^2 = z$	
$1 ⓦ^2$	3	$a^2 + 2ae + e^2 = ss$	
$3 - 2$	4	$2ae = ss - z$	
$2 - 4$	5	$a^2 - 2ae + e^2 = 2z - ss$	
$5\, ш^2$	6	$a - e = \overline{2z - ss}^{\frac{1}{2}}$	
$1 + 6$	7	$2a = s + \overline{2z - ss}^{\frac{1}{2}}$	
$7 \div 2$	8	$a = \frac{s + \overline{2z - ss}^{\frac{1}{2}}}{2}$	*For the greater Number sought.*
$1 - 6$	9	$2e = s - \overline{2z - ss}^{\frac{1}{2}}$	
$9 \div 2$	10	$e = \frac{s - \overline{2z - ss}^{\frac{1}{2}}}{2}$	*For the lesser Number sought.*

Ex. gr. If $s = 35$, and $z = 833$, then $a = \frac{s + \overline{2z - ss}^{\frac{1}{2}}}{2} = 28$, $e = \frac{s - \overline{2z - ss}^{\frac{1}{2}}}{2} = 7$.

PROB-

PROBLEM V.

550. To raise *Theorems* for finding any two Numbers a, e, from their Sum s, and Difference of their Squares x given.

Effection.

$$\begin{array}{r|r|l}
 & 1 & a+e=s \\
 & 2 & a^2-e^2=x \end{array}\Bigg\}\text{ By the Question.}$$

$$\begin{array}{r|r|l}
2\div 1 & 3 & a-e=\frac{x}{s} \\
1+3 & 4 & 2a=s+\frac{x}{s}=\frac{ss+x}{s} \\
4\div 2 & 5 & a=\frac{ss+x}{2s}\ \textit{The}\ \text{Theorem}\ \textit{for the greater Number.} \\
1-3 & 6 & 2e=s-\frac{x}{s}=\frac{ss-x}{s} \\
6\div 2 & 7 & e=\frac{ss-x}{2s}\ \textit{The}\ \text{Theorem}\ \textit{for the lesser Number.}
\end{array}$$

Ex. gr. If $s=35$, and $x=735$, then $a=\frac{ss+x}{2s}=28$, $e=\frac{ss-x}{2s}=7$.

PROBLEM. VI.

551. To raise *Theorems* for finding any two Numbers a, e, from their Sum s, and Sum of their Cubes m given.

Effection.

$$\begin{array}{r|r|l}
 & 1 & a+e=s \\
 & 2 & a^3+e^3=m \end{array}\Bigg\}\text{ By the Question.}$$

$$\begin{array}{r|r|l}
2\div 1 & 3 & a^2-ae+e^2=\frac{m}{s} \\
1-a & 4 & e=s-a \\
4\times a & 5 & ae=sa-aa \\
3+5 & 6 & a^2+e^2=\frac{m}{s}+sa-a^2 \\
4\odot^2 & 7 & e^2=s^2-2sa+a^2 \\
6-7 & 8 & a^2=\frac{m}{s}-s^2+3sa-2a^2
\end{array}$$

8 + $2a^2 - 3sa$

$8 + 2a^3 - 3sa$	9	$3a^2 - 3sa = \frac{m}{s} - ss = \frac{m - s^3}{s}$
$9 \div 3$	10	$a^2 - sa = \frac{m - s^3}{3s}$
$10 + \frac{ss}{4}$	11	$a^2 - sa + \frac{1}{4}ss = \frac{m - s^3}{3s} + \frac{ss}{4} = \frac{4m - s^3}{12s}$
$11\ w\ 2$	12	$a - \frac{1}{2}s = \left(\frac{4m - sss}{12s}\right)^{\frac{1}{2}} = \left(\frac{m}{3s} - \frac{ss}{12}\right)^{\frac{1}{2}}$
$12 + \frac{s}{2}$	13	$a = \left(\frac{4m - sss}{12s}\right)^{\frac{1}{2}} + \frac{1}{2}s$ *For the greater Number.*
$1 - 13$	14	$e = \frac{1}{2}s - \left(\frac{4m - sss}{12s}\right)^{\frac{1}{2}}$ *For the leſſer Number.*

Ex. gr. If $s = 35$, and $m = 22295$, then $a = \left(\frac{4m - sss}{12s}\right)^{\frac{1}{2}} + \frac{1}{2}s = 28$, and $e = \frac{1}{2}s - \left(\frac{4m - sss}{12s}\right)^{\frac{1}{2}} = 7$.

PROBLEM VII.

532. To raiſe *Theorems* for finding any two Numbers a, e, from their Sum s, and Difference of their Cubes n given.

Effection.

	1	$a + e = s$ *By the Queſtion.*
	2	$a^3 - e^3 = n$
$1 - a$	3	$e = s - a$
$3 \odot 3$	4	$e^3 = s^3 - 3s^2a + 3sa^2 - a^3$
$2 + 4$	5	$a^3 = n + s^3 - 3s^2a + 3sa^2 - a^3$
$\therefore$	6	$2a^3 - 3sa^2 + 3s^2a = n + s^3$ (In 430.)
$6 \div 2$	7	$a^3 - \frac{3}{2}sa^2 + \frac{3}{2}s^2a - \frac{n + s^3}{2} = 0$ *The* Theorem *for the greater Number.* (In 508.)
$1 - e$	8	$a = s - e$
$8 \odot 3$	9	$a^3 = s^3 - 3s^2e + 3se^2 - e^3$
$9 - 2$	10	$e^3 = s^3 - n - 3s^2e + 3se^2 - e^3$
$\therefore$	11	$2e^3 - 3se^2 + 3s^2e = s^3 - n$ (In 430.)
$11 \div 2$	12	$e^3 - \frac{3}{2}se^2 + \frac{3}{2}s^2e - \frac{s^3 - n}{2} = 0$ *The* Theorem *for the leſſer Number.* (In 508.)

Ex. gr. If $s=35$ and $n=21609$, then $a=28$, and $e=7=s-a$ Step. 3.

PROBLEM VIII.

553. To find any two Numbers a, e, from their Difference d, and Product p given.

Effection.

	1	$a-e=d$ } *By the Question.*
	2	$ae=p$ }
$1 \odot \overline{2}$	3	$a^2-2ae+e^2=dd$
2×4	4	$4ae=4p$
$3+4$	5	$a^2+2ae+e^2=dd-4p$
$5\,\omega^2$	6	$a+e=\overline{dd+4p}^{\frac{1}{2}}$
$6+1$	7	$2a=\overline{dd+4p}^{\frac{1}{2}}+d$
$7\div\overline{2}$	8	$a=\dfrac{\overline{dd+4p}^{\frac{1}{2}}+d}{2}$ Theorem I.
$6-1$	9	$2e=\overline{dd+4p}^{\frac{1}{2}}-d$
$9\div\overline{2}$	10	$e=\dfrac{\overline{dd+4p}^{\frac{1}{2}}-d}{2}$ Theorem II.

Ex. gr. If $d=21$, and $p=196$, then $a=\dfrac{\overline{dd+4p}^{\frac{1}{2}}+d}{2}=28$, and

$e=\dfrac{\overline{dd+4p}^{\frac{1}{2}}-d}{2}=7$.

PROBLEM IX.

554. To find any two Numbers a, e, from their Difference d, and Quotient q given.

Effection.

	1	$a-e=d$ } *By the Question.*
	2	$\dfrac{a}{e}=q$ }
$2\times e$	3	$a=qe$
$1+e$	4	$a=d+e$
3, 4,	5	$qe=d+e$
$5-e$	6	$qe-e=d$

$6\div\overline{q-1}$

$6 \div \overline{q-1}$	7	$e=\frac{d}{q-1}$	Theorem I.
2×7	8	$a=\frac{qd}{q-1}$	Theorem II.

Ex. gr. If $d=21$, and $q=4$, then $a=\frac{qd}{q-1}=28$, $e=\frac{d}{q-1}=7$.

Problem X.

555. To find any two Numbers a e, from their Difference d, and Sum of their Squares z given.

Effection.

	1	$a-e=d$	*By the Question.*
	2	$a^2+e^2=z$	
$1 \circledcirc^2$	3	$a^2-2ae+e^2=dd$	
$2-3$	4	$2ae \quad = z-dd$	
$2+4$	5	$aa+2ae+ee=2z-dd$	
$5 w^2$	6	$a+e=\overline{2z-dd}^{\frac{1}{2}}$	
$1+6$	7	$2a=\overline{2z-dd}^{\frac{1}{2}}+d$	
$7 \div \overline{2}$	8	$a=\frac{\overline{2z-dd}^{\frac{1}{2}}+d}{2}$	Theorem I.
$6-1$	9	$2e=\overline{2z-dd}^{\frac{1}{2}}-d$	
$9 \div \overline{2}$	10	$e=\frac{\overline{2z-dd}^{\frac{1}{2}}-d}{2}$	Theorem II.

Ex. gr. If $d=21$, and $z=833$, then $a=\frac{\overline{2z-dd}^{\frac{1}{2}}+d}{2}=28$, and $e=\frac{\overline{2z-dd}^{\frac{1}{2}}-d}{2}=7$.

Problem XI.

556. To find two Numbers a, e, from their Difference d, and Difference of their Squares x given.

Effection

Effection.

	1	$a-e=d$ } *By the Question.*
	2	$a^2-e^2=x$
$2\div 1$	3	$a+e=\frac{x}{d}$
$1+3$	4	$2a=d+\frac{x}{d}=\frac{dd+x}{d}$
$4\div 2$	5	$a=\frac{dd+x}{2d}$ Theorem I.
$3-5$	6	$e=\frac{x-dd}{2d}$ Theorem II.

Ex. gr. If $d=21$, and $x=735$, then $a=\frac{dd+x}{2d}=28$, and $e=\frac{x-dd}{2d}=7$.

PROBLEM XII.

557. To find two Numbers a, e, from their Difference d, and Sum of their Cubes m given.

Effection.

	1	$a-e=d$ } *By the Question.*
	2	$a^3+e^3=m$
$1-d+e$	3	$e=a-d$
3 ⓒ ^{3}	4	$e^3=a^3-3a^2d+3ad^2-d^3$
$2-4$	5	$a^3=m-a^3+3a^2d-3ad^2+d^3$
∴	6	$2a^3-3a^2d+3ad^2=m+d^3$
$6\div 2$	7	$a^3-\frac{3}{2}a^2d+\frac{3}{2}ad^2-\frac{m+d^3}{2}=0$ Theorem I.
$1+e$	8	$a=d+e$
8 ⓒ ^{3}	9	$a^3=d^3+3d^2e+3de^2+e^3$
$2-9$	10	$e^3=m-d^3-3d^2e-3de^2-e^3$
∴	11	$2e^3+3de^2+3d^2e=m-d^3$
$11\div 2$	12	$e^3+\frac{3}{2}de^2+\frac{3}{2}d^2e-\frac{m-d^3}{2}=0$ Theorem II.

Ex. gr. If $d=21$, and $m=22295$, then $a=28$, $e=7=a-d$ Step. 3.

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Problem. XIII.

558. To find two Numbers a, e, from their Difference d, and the Difference of their Cubes n given.

Effection.

	1	$a-e=d$	*By the Question.*
	2	$a^3-e^3=n$	
$2\div1$	3	$a^2+ae+e^2=\frac{n}{d}$	
$1-d+e$	4	$e=a-d$	
$4\times a$	5	$ae=aa-da$	
$3-5$	6	$a^2+e^2=\frac{n}{d}+da-aa$	
4 ⊙ ^{2}	7	$e^2=aa-2da+dd$	
$6-7$	8	$a^2=\frac{n}{d}-dd+3da-2a^2$	
∵	9	$3a^2-3da=\frac{n}{d}-dd=\frac{n-d^3}{d}$	
$9\div3$	10	$a^2-da=\frac{n-d^3}{3d}$	
$10+\frac{1}{4}dd$	11	$a^2-da+\frac{1}{4}dd=\frac{n-d^3}{3d}+\frac{1}{4}dd=\frac{4n-d^3}{12d}$	
$11\,\omega^2$	12	$a-\frac{1}{2}d=\left(\frac{4n-ddd}{12d}\right)^{\frac{1}{2}}=\left(\frac{n}{3d}-\frac{dd}{12}\right)^{\frac{1}{2}}$	
$12+\frac{1}{2}d$	13	$a=\left(\frac{4n-ddd}{12d}\right)^{\frac{1}{2}}+\frac{1}{2}d$	Theorem I.
$4,\ 13.$	14	$e=\left(\frac{4n-ddd}{12d}\right)^{\frac{1}{2}}-\frac{1}{2}d$	Theorem II.

Ex. gr. If $d=21$, and $n=21609$, then $a=28$, $e=a-d=7$ Step. 4.

Problem XIV.

559. To find any two Numbers a, e, from their Product p, and Quotient q given.

Effection.

	1	$ae=p$	By the Question.
	2	$\frac{a}{e}=q$	
1×2	3	$a^{2}=pq$	
$3w^{2}$	4	$a=\overline{pq}^{\frac{1}{2}}$	Theorem I.
$1\div2$	5	$ee=\frac{p}{q}$	
$5w^{2}$	6	$e=\overline{\frac{p}{q}}^{\frac{1}{2}}$	Theorem II.

Ex. gr. If $p=196$, and $q=4$, then $a=\overline{pq}^{\frac{1}{2}}=28$, $e=\overline{\frac{p}{q}}^{\frac{1}{2}}=7$.

Problem XV.

560. To find any two Numbers a, e, from their Product p, and Sum of their Squares z given.

Effection.

	1	$ae=p$	By the Question.
	2	$a^{2}+e^{2}=z$	
1×2	3	$2ae=2p$	
$2+3$	4	$a^{2}+2ae+e^{2}=z+2p$	
$4w^{2}$	5	$a+e=\overline{z+2p}^{\frac{1}{2}}$	
$2-3$	6	$a^{2}-2ae+e^{2}=z-2p$	
$6w^{2}$	7	$a-e=\overline{z-2p}^{\frac{1}{2}}$	
$5+7$	8	$2a=\overline{z+2p}^{\frac{1}{2}}+\overline{z-2p}^{\frac{1}{2}}$	
$8\div2$	9	$a=\frac{\overline{z+2p}^{\frac{1}{2}}+\overline{z-2p}^{\frac{1}{2}}}{2}$	Theorem I.
$5-7$	10	$2e=\overline{z+2p}^{\frac{1}{2}}-\overline{z-2p}^{\frac{1}{2}}$	
$10\div2$	11	$e=\frac{\overline{z+2p}^{\frac{1}{2}}-\overline{z-2p}^{\frac{1}{2}}}{2}$	Theorem II.

Ex.

Ex. gr. If $p=196$, and $x=833$, then $a=28$ (by Theorem I.) and $e=7$ (by Theorem II.)

Problem. XVI.

561. To find any two Numbers a, e, from their Product p, and Difference of their Squares x given.

Effection.

	1	$ae=p$ } *By the Question.*
	2	$a^2-e^2=x$ }
$1 \div a$	3	$e=\frac{p}{a}$
$3 \odot^2$	4	$ee=\frac{pp}{aa}$
$2+4$	5	$a^2=x+\frac{pp}{aa}$
$5 \times aa$	6	$a^4=a^2x+pp$
$6-a^2x$	7	$a^4-a^2x=pp$
$7+\frac{1}{4}xx$	8	$a^4-a^2x+\frac{1}{4}xx=pp+\frac{1}{4}xx=\frac{4pp+xx}{4}$
$8\,w^2$	9	$a^2-\frac{1}{2}x=\overline{\frac{4pp+xx}{4}}^{\frac{1}{2}}=\frac{\overline{4pp+xx}^{\frac{1}{2}}}{2}$
$9+\frac{1}{2}x$	10	$a^2=\frac{\overline{4pp+xx}^{\frac{1}{2}}+x}{2}$
$10\,w^2$	11	$a=\overline{\frac{\overline{4pp+xx}^{\frac{1}{2}}+x}{2}}^{\frac{1}{2}}$ Theorem I.
2.	12	$e^2=a^2-x=\frac{\overline{4pp+xx}^{\frac{1}{2}}-x}{2}$
$12\,w^2$	13	$e=\overline{\frac{\overline{4pp+xx}^{\frac{1}{2}}-x}{2}}^{\frac{1}{2}}$ Theorem II.

Ex. gr. If $p=196$, and $x=735$, then $a=28$ (by Theorem I.) *and* $e=7$ (by Theorem II.)

PRO-

Problem XVII.

562. To find two Numbers a, e, from their Product p, and Sum of their Cubes m given.

Effection.

	1	$ae=p$ } *By the Question.*
	2	$a^3+e^3=m$ }
$1\div a$	3	$e=\frac{p}{a}$
$3\,\text{ⓦ}^3$	4	$e^3=\frac{ppp}{aaa}$
$2-4$	5	$a^3=m-\frac{p^3}{a^3}=\frac{ma^3-p^3}{a^3}$
$5\times a^3$	6	$a^6=ma^3-p^3$
$6-ma^3$	7	$a^6-ma^3=-p^3$
$7+\frac{1}{4}mm$	8	$a^6-ma^3+\frac{1}{4}mm=\frac{1}{4}m^2-p^3=\frac{m^2-4p^3}{4}$
$8\,\omega^2$	9	$a^3-\frac{1}{2}m=\frac{\overline{m^2-4p^3}^{\frac{1}{2}}}{2}$
$9+\frac{1}{2}m$	10	$a^3=\frac{\overline{m^2-4ppp}^{\frac{1}{2}}+m}{2}$
$10\,\omega^3$	11	$a=\overline{\frac{\overline{mm-4ppp}^{\frac{1}{2}}+m}{2}}^{\frac{1}{3}}$ Theorem I.
$2-10$	12	$e^3=\frac{m-\overline{mm-4ppp}^{\frac{1}{2}}}{2}$
$12\,\omega^3$	13	$e=\overline{\frac{m-\overline{mm-4ppp}^{\frac{1}{2}}}{2}}^{\frac{1}{3}}$ Theorem II.

Ex. gr. If $p=196$, and $m=22295$, then $a=28$ (by Theorem I.) and $e=7$ (by Theorem II.)

Problem XVIII.

563. To find any two Numbers a, e, from their Product p, and Difference of their Cubes n given.

Effection.

Effection.

	1	$ae=p$ } *By the Question.*
	2	$a^3-e^3=n$ }
$1\div a$	3	$e=\frac{p}{a}$
$3 \odot^3$	4	$e^3=\frac{ppp}{aaa}$
$2+4$	5	$a^3=n+\frac{p^3}{a^3}=\frac{na^3+p^3}{a^3}$
$5\times a^3$	6	$a^6=na^3+p^3$
$6-na^3$	7	$a^6-na^3=p^3$
$7+\frac{1}{4}nn$	8	$a^6-na^3+\frac{1}{4}nn=p^3+\frac{1}{4}nn=\frac{4p^3+nn}{4}$
$8w^2$	9	$a^3-\frac{1}{2}n=\frac{\overline{4p^3+nn}\vert^{\frac{1}{2}}}{2}$
$9+\frac{1}{2}n$	10	$a^3=\frac{\overline{4p^3+nn}\vert^{\frac{1}{2}}+n}{2}$
$10w^3$	11	$a=\overline{\frac{\overline{4ppp+nn}\vert^{\frac{1}{2}}+n}{2}}\Big\vert^{\frac{1}{3}}$ Theorem I.
2,	12	$e^3=a^3-n=\frac{\overline{4ppp+nn}\vert^{\frac{1}{2}}-n}{2}$
$12w^3$	13	$e=\overline{\frac{\overline{4ppp+nn}\vert^{\frac{1}{2}}-n}{2}}\Big\vert^{\frac{1}{3}}$ Theorem II.

Ex. gr. If $p=196$, and $n=21609$, then $a=28$ (by Theorem I.) and $e=7$ (by Theorem II.)

PROBLEM XIX.

564. To find any two Numbers a, e, from their Quotient q, and the Sum of their Squares z given.

Effection.

	1	$\frac{a}{e}=q$ } By the Question.
	2	$a^2+e^2=z$ }
$1\times e$	3	$eq=a$
$3\odot^{2}$	4	$e^2q^2=a^2$
$2-a^2$	5	$e^2=z-a^2$
$4+5$	6	$e^2q^2+e^2=z$
$6\div\overline{qq+1}$	7	$e^2=\frac{z}{qq+1}$
$z-7$	8	$a^2=z-\frac{z}{qq+1}=\frac{qqz}{qq+1}$
$8\,ww^{2}$	9	$a=q\times\left(\frac{z}{qq+1}\right)^{\frac{1}{2}}=\left(\frac{qqz}{qq+1}\right)^{\frac{1}{2}}$ Theorem I.
$7\,ww^{2}$	10	$e=\left(\frac{z}{qq+1}\right)^{\frac{1}{2}}$ Theorem II.

Ex. gr. If $q=4$, and $z=833$, then $a=28$ (by Theorem I.) and $e=7$ (by Theorem II.)

PROBLEM XX.

565. To find any two Numbers a, e, from their Quotient q, and the Difference of their Squares x given.

Effection.

	1	$\frac{a}{e}=q$ } By the Question.
	2	$a^2-e^2=x$ }
$1\times e$	3	$a=qe$
3×2	4	$aa=qqee$
$2+ee$	5	$aa=x+ee$
4, 5.	6	$qqee=x+ee$
Whence	7	$ee=\frac{x}{qq-1}$
$2+8$	8	$aa=x+\frac{x}{qq-1}=\frac{qqx}{qq-1}$

$8w^{2}$	9	$a=q\left(\dfrac{x}{qq-1}\right)^{\frac{1}{2}}$	Theorem I.
$7w^{2}$	10	$e=\left(\dfrac{x}{qq-1}\right)^{\frac{1}{2}}$	Theorem II.

Ex. gr. If $q=4$, and $x=735$, then $a=28$ (by Theorem I.) and $e=7$ (by Theorem II.)

PROBLEM XXI.

566. To find any two Numbers a, e, from their Quotient q, and the Sum of their Cubes m given.

Effection.

	1	$\dfrac{a}{e}=q$	By the Question.
	2	$a^{3}+e^{3}=m$	
$1\times e$	3	$eq=a$	
$3\div q$	4	$e=\dfrac{a}{q}$	
$4\odot^{3}$	5	$e^{3}=\dfrac{a^{3}}{q^{3}}$	
$2-3$	6	$a^{3}=m-\dfrac{a^{3}}{q^{3}}=\dfrac{mqqq-aaa}{qqq}$	
$6\times qqq$	7	$q^{3}a^{3}=mq^{3}-a^{3}$	
$7+a^{3}$	8	$q^{3}a^{3}+a^{3}=mq^{3}$	
$8\div \overline{qqq+1}$	9	$a^{3}=\dfrac{q^{3}m}{q^{3}+1}$	
$9w^{3}$	10	$a=q\left(\dfrac{m}{qqq+1}\right)^{\frac{1}{3}}$	Theorem I.
$2-9$	11	$e^{3}=m-\dfrac{qqqm}{qqq+1}=\dfrac{m}{qqq+1}$	
$11w^{3}$	12	$e=\left(\dfrac{m}{qqq+1}\right)^{\frac{1}{3}}$	Theorem II.

Ex. gr. If $q=4$, and $m=22295$, then $a=28$ (by Theorem I.) and $e=7$ (by Theorem II.)

PRO-

PROBLEM XXII.

567. To find any two Numbers a, e, from their Quotient q, and the Difference of their Cubes n given.

Effection.

	1	$\frac{a}{e}=q$ } *By the Question.*
	2	$aaa-eee=n$
$1\times e$	3	$eq=a$
$3\div q$	4	$e=\frac{a}{q}$
$4\odot 3$	5	$e^3=\frac{aaa}{qqq}$
$2+5$	6	$aaa=n+\frac{aaa}{qqq}=\frac{nqqq+aaa}{qqq}$
Whence	7	$a=q\times\sqrt[3]{\frac{n}{qqq-1}}$ Theorem I.
And	8	$e=\sqrt[3]{\frac{n}{qqq-1}}$ Theorem II.

Ex. gr. If $q=4$, and $n=21609$, then $a=28$ (Theorem I.) and $e=7$ (Theorem II.)

PROBLEM XXIII.

568. To find any two Numbers a, e, from the Sum of their Squares z, and the Difference of their Squares x given.

Effection.

	1	$aa+ee=z$ } *By the Question.*
	2	$aa-ee=x$
$1+2$	3	$2aa=z+x$
$3\div 2$	4	$aa=\frac{z+x}{2}$
$1-2$	5	$2ee=z-x$
$5\div 2$	6	$ee=\frac{z-x}{2}$

$4w^2$	7	$a=\sqrt{\frac{z+x}{2}}$ Theorem I.
$6w^2$	8	$e=\sqrt{\frac{z-x}{2}}$ Theorem II.

Ex. gr. If $z=833$, and $x=735$; then $a=28$ (Theorem I.) and $e=7$ Theorem II.

Problem XXIV.

569. To find any two Numbers a, e, from the Sum of their Squares z, and Sum of their Cubes m given.

Effection.

Make $a+e=2y$, and $a-e=2d$,

Then $\begin{cases} 1 & y+u=a \\ 2 & y-u=e \end{cases}$ (In. 546.)

$1 ⊙^2$	3	$y^2+2yu+uu=aa$
$2 ⊙^2$	4	$y^2-2yu+uu=ee$
$1 ⊙^3$	5	$y^3+3y^2u+3yu^2+u^3=aaa$
$2 ⊙^3$	6	$y^3-3y^2u+3yu^2-u^3=eee$
$3+4$	7	$2y^2+2u^2=a^2+e^2=z$ By the Question.
$5+6$	8	$2y^3+6yu^2=a^3+e^3=m$ By the Question.
$7-2yy$	9	$2u^2=z-2y^2$
$9\div2$	10	$u^2=\frac{z-2yy}{2}$
$8-2yyy$	11	$6yu^2=m-2y^3$
$11\div6y$	12	$u^2=\frac{m-2yyy}{6y}$
10, 12	13	$\frac{z-2yy}{2}=\frac{m-2yyy}{6y}$
$13\times6y$	14	$3zy-6yyy=m-2yyy$.
$14+2yyy$	15	$3zy-4yyy=m$
$15\div4$	16	$-yyy+\frac{3}{4}zy-\frac{m}{4}=0$ Theorem I.
$10w^2$	17	$u=\sqrt{\frac{z-2yy}{2}}$ Theorem II.

Ex. gr. If $z=833$, and $n=22295$, then $y=17.5$ (Theorem I.) and $u=\left(\frac{z-2yy}{2}\right)^{\frac{1}{2}}=10.5$ (Theorem II.) consequently $a=y+u=28$, and $e=y-u=7$.

Problem XXV.

570. To find any two Numbers a, e, from the Sum of their Squares z, and Difference of their Cubes n given.

Effection.

Things being represented as in the last.

	1	$2y^2+2u^2=a^2+e^2=z$	By the Question.
	2	$2uuu+6uy^2=a^3-e^3=n$	
$1-2uu$	3	$2yy=z-2uu$	
$3\div2$	4	$yy=\frac{z-2uu}{2}$	
$2-2uuu$	5	$6yyu=n-2uuu$	
$5\div6u$	6	$yy=\frac{n-2uuu}{6u}$	
4, 6,	7	$\frac{z-2uu}{2}=\frac{n-2uuu}{6u}$	
$7\times6u$	8	$3uz-6u^3=n-2u^3$	
$8+2uuu$	9	$3uz-4u^3=n$	
$9\div4$	10	$-u^3+\frac{3}{4}zu-\frac{n}{4}=0$	Theorem I.
$4\omega^{\frac{1}{2}}$	11	$y=\left(\frac{z-2uu}{2}\right)^{\frac{1}{2}}$	Theorem II.

Ex. gr. If $z=833$, and $n=21609$, then $u=10.5$ (Theorem I.) and $y=\left(\frac{z-2uu}{2}\right)^{\frac{1}{2}}=17.5$ (Theorem II.) consequently $a=y+u=28$, and $e=y-u=7$, by the Hypothesis.

Problem XXVI.

571. To find any two Numbers a, e, from the Difference of their Squares x, and the Sum of their Cubes m given.

Effection

Effection.

Things being again represented as before.

	1	$4yu=aa-ee=x$ } *By the Question.*
	2	$2y^3+6yuu=d^3+e^3=m$
$1\div 4y$	3	$u=\frac{x}{4y}$
$3\odot^2$	4	$uu=\frac{xx}{16yy}$
$2-2yyy$	5	$6yuu=m-2y^3$
$5\div 6y$	6	$uu=\frac{m-2yyy}{6y}$
4, 6.	7	$\frac{xx}{16yy}=\frac{m-2yyy}{6y}$
$7\times 48yy$	8	$3xx=8my-16yyyy$
$8\div 16$	9	$-yyyy+\frac{1}{2}my-\frac{3xx}{16}=0$ Theorem I.
3.	10	$u=\frac{x}{4y}$ Theorem II.

Ex. gr. If $x=735$, and $m=22295$, then $y=17.5$, (Theorem I.) and $u=10.5$ (Theorem II.) consequently $a=y+u=28$, and $e=y-u=7$.

PROBLEM. XXVII.

572. To find any two Numbers a, e, from the Difference of their Squares x, and the Difference of their Cubes n given.

Effection.

Things being once more represented as above.

	1	$4yu=a^2-e^2=x$ } *By the Question.*
	2	$2uuu+6uy^2=a^3-e^3=n$
$1\div 4u$	3	$y=\frac{x}{4u}$
$3\odot^2$	4	$yy=\frac{xx}{16uu}$
$2-2uuu$	5	$6uyy=n-2uuu$
$5\div 6u$	6	$yy=\frac{n-2uuu}{6u}$

4, 6.

4, 6.	7	$\frac{xx}{16uu}=\frac{n-2uuu}{6u}$	
$7\times48uu$	8	$3xx=8un-16uuuu$	
$8\div\overline{16}$	9	[illegible] $+\frac{1}{2}nu-\frac{3xx}{16}=0$	Theorem I.
3.	10	$y=\frac{z}{4u}$	Theorem II.

Ex. gr. If $x=735$, and $n=21609$, then $u=10.5$ (Theorem I.) and $y=17.5$ (Theorem II.) coonsequently $a=u+y=28$, $\cancel{e=y-u}=7$.

Problem XXVIII.

573. To find any two Numbers a, e, from the Sum of their Cubes m, and Difference of their Cubes n given.

Effection.

	1	$a^3+e^3=m$	By the Question
	2	$a^3-e^3=n$	
$1+2$	3	$2a^3=m+n$	
$3\div\overline{2}$	4	$a^3=\frac{m+n}{2}$	
$1-2$	5	$2e^3=m-n$	
$5\div\overline{2}$	6	$e^3=\frac{m-n}{2}$	
$4w^3$	7	$a=\overline{\frac{m+n}{2}}^{\frac{1}{3}}$	Theorem I.
$6w^3$	8	$e=\overline{\frac{m-n}{2}}^{\frac{1}{3}}$	Theorem II.

Ex. gr. If $m=22295$, and $n=21609$, then $a=28$ (Theorem I.) $e=7$ (Theorem II.)

Scoleum I.

574. *Note*, All the foregoing Problems of this Chapter are to be seen in Dr. *Pell's Algebra*, except those wherein m and n is concerned.

Problem. XXIX.

575. To find three Numbers a, e, y, from the Sum of the Cubes of every two of them given.

Effection.

Effection.

	1	$a^3+e^3=b$ } By the Question.
	2	$a^3+y^3=c$
	3	$e^3+y^3=d$
$1-2$	4	$e^3-y^3=b-c$
$4+yyy$	5	$e^3=b-c+y^3$
$3-yyy$	6	$e^3=d-y^3$
5, 6,	7	$b-c+y^3=d-y^3$
∴	8	$2yyy=d-b+c$
$8\div 2$	9	$y^3=\frac{d-b+c}{2}$
$3-9$	10	$e^3=d-\frac{d-b+c}{2}=\frac{d+b-c}{2}$
$2-10$	11	$a^3=b-\frac{d+b-c}{2}=\frac{b-d+c}{2}$
$11\,uw^3$	12	$a=\sqrt[3]{\frac{b-d+c}{2}}$ Theorem I.
$10\,uw^3$	13	$e=\sqrt[3]{\frac{d+b-c}{2}}$ Theorem II.
$9\,uw^3$	14	$y=\sqrt[3]{\frac{d-b+c}{2}}$ Theorem III.

Ex. gr. If $b=1072$, $c=854$, and $d=468$, then $a=9$ (Theo. I.) $e=7$ (Theo. II.) $y=5$ (Theo. III.)

PROBLEM XXX.

576. What four Numbers are thofe a, e, y, u, whofe Sum is s, and the Difference of the firft lefs the fecond is b, of the fecond lefs the third c, and of the third lefs the fourth d?

Effection.

	1	$a+e+y+u=s$ } By the Question.
	2	$a-e=b$
	3	$e-y=c$
	4	$y-u=d$
$1-e-y-u$	5	$a=s-e-y-u$
$2+e$	6	$a=b+e$

6, 5	7	$b+e=s-e-y-u$	
Whence	8	$e=\frac{s-b-y-u}{2}$	
3+4	9	$e=c+y$	
9, 8	10	$c+y=\frac{s-b-y-u}{2}$	
Whence	11	$y=\frac{s-b-2c-u}{3}$	
4+u	12	$y=d+u$	
12, 11	13	$d+u=\frac{s-b-2c-u}{3}$	
Whence	14	$u=\frac{s-b-2c-3d}{4}$	Theorem I.
12,	15	$y=d+u=\frac{s+d-b-2c}{4}$	Theorem II.
9,	16	$e=c+y=\frac{s+2c+d-b}{4}$	Theorem III.
6,	17	$a=b+e=\frac{s+3b+2c+d}{4}$	Theorem IV.

Ex. gr. If $s=37$, $b=3$, $c=1$, and $d=4$, then $u=5$ (Theorem I.) $y=9$ (Theorem II.) $e=10$ (Theorem III.) $a=13$ (Theorem IV.) Which will be an Anſwer to the following Queſtion.

Suppoſe four Hedgers to earn amongſt them 37 Pounds; of which the firſt earn'd 3 *l.* more than the ſecond; the ſecond 1 *l.* more than the third, and the third 4 *l.* more than the fourth: It is required to find the Earnings of each.

Problem XXXI.

577. From any two Terms given, of three Terms (*A*, *B*, *C*) in *Arithmetical Proportion*, to find the third.

Effection.

				Given	Sought
	1	$A+C=2B$	(In. 176.)		
1−C	2	$A=2B-C$	Theorem I.	B, C,	A,
$\overline{1}\div 2$	3	$B=\frac{A+C}{2}$	Theorem II.	A, C,	B,
1−A	4	$C=2B-A$	Theorem III.	A, B.	C.

Problem XXXII.

578. From any three Terms given, of four Terms (A, B, C, D) in *Arithmetical Proportion*, to find the fourth.

Effection.

				Given	Sought
	1	$A+D=B+C$	(In. 174)		
$1-D$	2	$A=B+C-D$	Theorem I.	B, C, D,	A,
$1-C$	3	$B=A+D-C$	Theorem II.	A, C, D,	B,
$1-B$	4	$C=A+D-B$	Theorem III.	A, B, D,	C,
$1-A$	5	$D=B+C-A$	Theorem IV.	A, C, B,	D,

Problem XXXIII.

579. From any two Terms given, of three Terms (A, B, C) *Simple Geometrical Proportion Direct*, to find the third.

Effection.

				Given	Sought
	1	$AC=BB$	(In. 191.)		
$1\div C$	2	$A=\frac{BB}{C}$	Theorem I.	B, C,	A,
$1\,w^{2}$	3	$B=\overline{AC}^{\frac{1}{2}}$	Theorem II.	A, C,	B,
$1\div A$	4	$C=\frac{BB}{A}$	Theorem III.	A, B,	D,

Problem XXXIV.

580. From any three Terms given, of four Terms (A, B, C, D) in *Simple Geometrical Proportion Direct*, to find the fourth.

Effection.

				Given	Sought
	1	$AD=BC$	(In. 189.)		
$1\div D$	2	$A=\frac{BC}{D}$	Theorem I.	B, C, D,	A,
$1\div C$	3	$B=\frac{AD}{C}$	Theorem II.	A, C, D,	B,
$1\div B$	4	$C=\frac{AD}{B}$	Theorem III.	A, B, D,	C,
$1\div A$	5	$D=\frac{BC}{A}$	Theorem IV.	A, B, C,	D,

Pro-

Problem XXXV.

581. From any five Terms given, of six in *Compound Geometrical Proportion Direct*, to find the sixth. *Ex. gr.* Suppose C in Υ *Time* to have the same Ratio to P in T *Time*, that R has to Υ, *i. e.* $C\Upsilon : PT = R : I$ (In. 198.) Then

Effection.

				Given	Sought
	1	$C\Upsilon I = PTR$	(In. 189)		
$1 \div C\Upsilon$	2	$I = \frac{PTR}{C\Upsilon}$	Theo. I.	$C, P, R, T, \Upsilon,$	$I,$
$1 \div CI$	3	$\Upsilon = \frac{PTR}{CI}$	Theo. II.	$C, I, P, R, T,$	$\Upsilon,$
$1 \div \Upsilon I$	4	$C = \frac{PTR}{\Upsilon I}$	Theo. III.	$I, P, R, T, \Upsilon,$	$C,$
$1 \div TR$	5	$P = \frac{C\Upsilon I}{TR}$	Theo. IV.	$C, I, R, T, \Upsilon,$	$P,$
$1 \div PR$	6	$T = \frac{C\Upsilon I}{PR}$	Theo. V.	$C, I, P, R, \Upsilon,$	$T,$
$1 \div PT$	7	$R = \frac{C\Upsilon I}{PT}$	Theo. VI.	$C, I, P, T, \Upsilon,$	$R,$

Problem XXXVI.

582. From any two Terms given, of three Terms (A, B, C) in *Harmonical Proportion*, to find the third.

Effection.

				Given	Sought
	1	$\frac{C}{A} = \frac{C - B}{B - A}$	(In. 202.)		
$1 \times A$	2	$C = \frac{CA - BA}{B - A}$			
$2 \times \overline{B - A}$	3	$BC - AC = CA - BA$			
$3 + AC$	4	$BC = 2CA - BA$			
$4 \div \overline{2C - B}$	5	$\frac{BC}{2C - B} = A$	Theorem I.	$B, C,$	$A,$
$4 + AB$	6	$BC + BA = 2CA$			
$6 \div \overline{A + C}$	7	$B = \frac{2CA}{A + C}$	Theorem II.	$A, C,$	$B,$
$6 - BC$	8	$AB = 2AC - BC$			
$8 \div \overline{2A - B}$	9	$\frac{AB}{2A - B} = C$	Theorem II.	$A, B,$	$C,$

Pro-

PROBLEM. XXXVII.

583. From any three Terms given, of four Terms (A, B, C, D) in *Harmonical Proportion*, to find the fourth.

Effection.

			Given	Sought
	1	$\frac{D}{A}=\frac{D-C}{B-A}$ (In. 202.)		
$1\times A\times\overline{B-A}$	2	$DB-DA=DA-CA$		
$2+DA$	3	$DB=2DA-CA$		
$3\div\overline{2D-C}$	4	$\frac{DB}{2D-C}=A$ Theo. I.	B, C, D,	A,
$3\div D$	5	$B=\frac{2DA-CA}{D}$ Theo. II.	A, C, D,	B,
$3+AC$	6	$DB+CA=2DA$		
$6-DB$	7	$CA=2DA-DB$		
$7\div A$	8	$C=\frac{2DA-DB}{A}$ Theo. III.	A, B, D,	C,
$7\div 2\overline{A-B}$	9	$\frac{CA}{2A-B}=D$ Theo. IV.	A, B, C,	D.

PROBLEM XXVIII.

584. From any two Terms given, of three Terms (A, B, C) in *Contra-Harmonical Proportion*, to find the third.

Effection.

			Given	Sought
	1	$\frac{C}{A}=\frac{B-A}{C-B}$ (In. 203.)		
$1\times A\times\overline{C-B}$	2	$CC-CB=BA-AA.$		
Or.	3	$AA-BA=BC-CC$		
Whence	4	$A=\overline{\frac{1}{4}BB-CC+BC}^{\frac{1}{2}}+\frac{1}{2}B$ Theo. I.	B, C,	A,
$2+AA+CB$	5	$CC+AA=AB+CB$		
$5\div\overline{A+C}$	6	$\frac{AA+CC}{A+C}=B$ Theo. II.	A, C,	B,
$2+\frac{1}{4}BB$	7	$CC-CB+\frac{1}{4}BB=BA-AA+\frac{1}{4}BB$		
Whence	8	$C=\overline{BA-AA+\frac{1}{4}BB}^{\frac{1}{2}}+\frac{1}{2}B$ Theo. III.	A, B,	C,

PROBLEM. XXXIX.

585. To divide a given Number g into any propofed Number of Parts or Shares, which fhall be proportional to certain Numbers given, (Ex. gr. b, c, d, f.) whofe Sum equals s, *i. e.*

Effection.

By the Queſtion.	1	$a:b=e:c$
	2	$e:c=y:d$
	3	$y:d=u:f$ to find a, e, y, u.
	4	$a+e+y+u=g$
	5	$b+c+d+f=s$
From Step. 1.	6	$a=\frac{be}{c}$ (In. 579.)
From Step. 4.	7	$a=g-e-y-u$
6, 7,	8	$\frac{be}{c}=g-e-y-u$
$8\times c$	9	$be=cg-ce-cy-cu$
Whence	10	$e=\frac{cg-cy-cu}{b+c}=c\times\frac{g-y-u}{b+c}$
Step. 2.	11	$e=\frac{cy}{d}$ (In. 579.)
11, 10.	12	$\frac{cy}{d}=\frac{cg-cy-cu}{b+c}$
Whence	13	$y=d\times\frac{g-u}{b+d+c}$
Step. 3.	14	$y=\frac{du}{f}$
14, 13.	15	$\frac{du}{f}=d\times\frac{g-u}{b+d+c}$
Whence	16	$u=\frac{fg}{b+c+d+f}$
But	17	$b+c+d+f=s$ Step. 5.
$\therefore$	18	$u=\frac{fg}{s}$, or $s:f=g:u$
$\therefore$ 14,	19	$y=\frac{dg}{s}$, or $s:d=g:y$

∴	11	20	$e=\frac{cg}{s}$, or $s:c=g:e$
∴	6	21	$a=\frac{bg}{s}$, or $s:b=g:a$

Which gives this *Theorem* for resolving Questions in *Simple Fellowship*, or *Fellowship* without Time.

As the joint *Stock* s of any Number of Partners in Trafick is to their *whole Gain* or *Loss*, so is each Man's *particular Share* of the *Stock* to his particular *Share* in the *Gain* or *Loss*. Which Analogy may be most conveniently wrought by this *Rule*.

Divide the whole Gain or Loss g by the whole Stock s, and the Quotient $\frac{g}{s}$ multiplied into each Man's particular Share of the Stock, will produce his particular Share in the Gain or Loss.

Ex. gr. Suppose three Partners (*A*, *B*, and *C*) make a joint Stock of 1403 *l.* $=s$; of which *A* puts in 530 *l.* 10 *s.* $=b$, *B* puts in 462 *l.* 16 *s.* $=c$, *C* puts in 409 *l.* 14 *s.* $=d$: With this Stock they trade a certain Time, and gain 731 *l.* 15 *s.* $=g$: It is required to determine every Man's *particular Share* in that *Gain* proportional to his *Part* in the *Stock*.

Here $\frac{g}{s}=\frac{731.75}{1403}=0.521561$ near, Then

		l.		*l.*	*s.*	*d.*	*qrs.*	
$\frac{bg}{s}=530.5$		276.6881	or	276.	13.	09.	0.576.	for *A*'s *Gain.*
$\frac{cg}{s}=462.8$	$\times 0.521561=$	241.3784	or	241.	07.	06.	3.264.	for *B*'s *Gain.*
$\frac{dg}{s}=409.7$		213.6835	or	213.	13.	08.	0.162.	for *C*'s *Gain.*
	Proof	731.75	or	731.	15.			= the *whole Gain.*

And the same *Theorem* serves for solving Questions in *Compound Fellowship*, or *Fellowship* with Time, if we multiply each Man's particular Stock into the Time of its Continuance, according to the Rule of Compound Proportion (In. 581.) calling these particular Products b, c, d, *&c.* and their Sums s.

Ex. gr. Four Merchants (*A*, *B*, *C*, and *D*) compose a *joint Stock*, as follows.

l. *Months.*

$$\left.\begin{matrix} A \\ B \\ C \\ D \end{matrix}\right\} \text{puts in} \left\{\begin{matrix} l. \\ 100 \\ 80 \\ 75 \\ 50 \end{matrix}\right\} \text{for} \left\{\begin{matrix} Months. \\ 5 \\ 9 \\ 8 \\ 12 \end{matrix}\right\}$$ With which they trade and gain 217 *l.* 16 *s.* or 217.8 *l.*

It is required to determine each Man's *Share* of that *Gain.*

Here $b=100\times5=500$, $c=80\times9=720$; $d=75\times8=600$, $f=50\times12=600$: consequently $b+c+d+f=2420=s$: and $\frac{217.8}{2420}=0.09=g$. Then

$$\left.\begin{matrix} \frac{bg}{s}=500 \\ \frac{cg}{s}=720 \\ \frac{dg}{s}=600 \\ \frac{fg}{s}=600 \end{matrix}\right\}\times0.09=\left\{\begin{matrix} 45\ l. & \text{for } A\text{'s Gain.} \\ 64.8\ l. \text{ or } 64\ l.\ 16\ s. & \text{for } B\text{'s Gain.} \\ 54\ l. & \text{for } C\text{'s Gain.} \\ 54\ l. & \text{for } D\text{'s Gain.} \end{matrix}\right.$$

Proof 217.8 *l.* or 217 *l.* 16 *s.* = the *whole Gain.*

Problem. XL.

586. Suppose several Sums of Money (*viz.* b, c, d, *&c.* whose Sum Total $b+c+d$ *&c.* $=s$) due at *several Times*, *viz.* b in f *Time*, c in g *Time*, d in h *Time*, *&c.* It is required to find a *mean equated Time* a, wherein the whole may be paid without Damage to Debtor or Creditor.

Effection.

	1	$s:f=b:\frac{bf}{s}$
	2	$s:g=c:\frac{cg}{s}$
	3	$s:h=d:\frac{dh}{s}$
		&c. &c. &c.
1+2+3 *&c.*	4	$a=\frac{bf+cg+dh\ \&c.}{b+c+d\ \&c.}$

Whence

Whence we have this Theorem for the *Equation of Payments*.

Multiply each particular Sum into its respective Time, and divide the Sum of the Products by the whole Debt; the Quotient will be the *equated* Time required.

Example 1. *A* is indebted to *B* 1000 *l.* whereof he is to pay 600 *l.* in 4 Months, 300 *l.* in 6 Months, and 100 *l.* in 9 Months; but they agree to make an *equated mean Time* for the *Payment* of the whole. What is that Time?

Here $b=600$, $c=300$, $d=100$, $f=4$, $g=6$, $h=9$, $1000=s$: Whence $a=\frac{bf+cg+dh}{b+c+d}=5\frac{1}{10}$ Months.

Example 2. *A* is indebted to *B* 640 *l.* whereof he is to pay 40 *l.* present Money, 350 *l.* in 3 Months, and the rest (which is 250 *l.*) in 8 Months; and they agree to make an *equated Time* for the whole *Payment*. What is that Time?

Here $b=40$, $c=350$, $d=250$, $f=0$, $g=3$, $h=8$, $640=s$: Whence $a=\frac{bf+cg+dh}{b+c+d}=4\frac{41}{64}$ Months.

Problem XLI.

587. From the particular Quantities of the *Ingredients* which go to compose any *Mixture*, with their particular Rates given, to find the *Mean Rate* or *Price* of the *Mixture*.

This is what Arithmeticians call *Alligation Medial*.

Ex. gr. A Vintner mixes $b=31\frac{1}{2}$ Gallons of *Malaga*, worth $p=7$ *s.* 6 *d.* the Gallon; with $c=18$ Gallons of *Canary*, at $q=6$ *s.* 9 *d.* the gallon; with $d=13\frac{1}{2}$ Gallons of *Sherry*, at $r=5$ *s.* the Gallon, and $f=27$ Gallons of *White*, at $s=4$ *s.* 3 *d.* the Gallon. What is the Price a of a Gallon of this Mixture?

$$\left.\begin{array}{l} 1:p=b:bp \\ 1:q=c:cq \\ 1:r=d:dr \\ 1:s=f:fs \end{array}\right\}\text{the Price of the}\left\{\begin{array}{l} \textit{Malaga} \\ \textit{Canary} \\ \textit{Sherry} \\ \textit{White} \end{array}\right.$$

Then $b+c+d+f:bp+cq+dr+fs=1:a$; whence $a=\frac{bp+cq+dr+fs}{b+c+d+f}=6$ Shillings, which gives the following *Theorem*.

Multiply the Quantity of each Ingredient into its particular Rate, and the Sum of all these Products divided by the Sum of the Quantities of all the Ingredients will give the mean Rate required.

 Pro-

PROBLEM XLII.

588. *So to compound unlike Mixtures of two or more different Ingredients, that the Ingredients may have a given Proportion to one another.*

Ex. gr. Suppose three Mixtures of Metal, of the first of which a Pound Averdupois contains 12 ℥ of Silver, 1 ℥ of Brass, and 3 ℥ of Tin; of the second a Pound contains 1 ℥ of Silver, 12 ℥ of Brass, 3 ℥ of Tin; and a Pound of the third contains 14 ℥ of Brass and 2 ℥ of Tin: It is required to make a Composition of these Mixtures, a Pound of which may contain 4 ℥ of Silver, 9 ℥ of Brass, and 3 ℥ of Tin. Otherwise, making $b=12$, $d=1$, $f=3$, $g=1$, $h=12$, $k=3$, $l=0$, $m=14$, $n=2$, $p=4$, $q=9$, $r=3$, there is given.

$$\left.\begin{array}{r}\text{One}\\ \text{A second}\\ \text{A third}\end{array}\right\}\text{Mixture containing}\left\{\begin{array}{ccccc}\textit{Sil.} & & \textit{Br.} & & \textit{Tin.}\\ b & + & d & + & f\\ g & + & h & + & k\\ l & + & m & + & n\end{array}\right.$$

Out of which it is required to make a Composition, a Pound of which may contain p Silver $+p$ Brass $+\ r$ Tin. Vide *Newton's Algebra*, p. 75.

Effection.

For the Parts of a Pound which are required of the first Mixture put a, of the second e, and of the third y.

$$\text{Then}\left\{\begin{array}{l}ab+ad+af\\ eg+eh+ek\\ yl+ym+yn\end{array}\right\}=p+q+r,\ \text{by the Question.}$$

$$\text{Also}\left\{\begin{array}{r|l}1 & ab+eg+yl=p\\ 2 & ad+eh+ym=q\\ 3 & af+ek+yn=r\end{array}\right\}\textit{By the Question.}$$

$$\begin{array}{r r|l}1, & 4 & a=\dfrac{p-eg-yl}{b}\\ 2, & 5 & a=\dfrac{q-eh-ym}{d}\\ 3, & 6 & a=\dfrac{r-ek-yn}{f}\\ \text{Whence} & 7 & e=\dfrac{dp-bq+\overline{bm-dl}\times y}{dg-bh}=\dfrac{fq-dr+\overline{dn-fm}\times y}{fh-dk}\end{array}$$

Or

Or by making | 8 | $A=dp-bq$, $B=bm-dl$, $G=dg-bb$, $D=fg-dr$, $Z=dn-fm$, $T=fb-dk$, (i. e. $A=-104$, $B=+168$, $G=-143$, $D=+24$, $Z=-40$, $T=+33$) $e=\frac{A+By}{G}=\frac{D+Zy}{T}$

$\therefore$ | 9 | $y=\frac{TA-GD}{GZ-BT}=0$, $e=\frac{ZA-BD}{GZ-BT}=\frac{8}{11}$, $a=\frac{p-eg-yl}{b}=\frac{3}{11}$.

Therefore the Compoſition muſt be made up of $\frac{3}{11}$ ℔ of the firſt Mixture, $\frac{8}{11}$ ℔ of the ſecond, and none of the third.

Problem XLIII.

589. The *Prices* of *ſeveral Mixtures* of the *ſame Ingredients*, and the *Proportion* of the *Ingredients* mixed together being given, to determine the *Price* of the *Things mixed*.

Ex. gr. One bought $40=b$ Buſhels of Wheat, $24=g$ Buſhels of Barley, and $20=l$ Buſhels of Oats together for $15.6\,l.=p$: Again, he bought of the ſame Grain, $26=d$ Buſhels of Wheat, $30=b$ Buſhels of Barley, and $50=m$ Buſhels of Oats together for $16\,l.=q$: And thirdly, he bought of the like Kind of Grain $24=f$ Buſhels of Wheat, $120=k$ Buſhels of Barley, and $100=n$ Buſhels of Oats together for $34\,l.=r$. It is demanded at what Rate a Buſhel of each of theſe Grains ought to be valued? *Newton*'s *Algebra*, p. 77.

Here putting a for the Price of the Buſhel of Wheat, e of Barley, and y of Oats.

$$\text{Then}\left\{\begin{array}{l} ab+eg+yl=p \\ ad+eb+ym=q \\ af+ek+yn=r \end{array}\right\}\text{as in the laſt.}$$

When ſubſtituting exactly as before, there will come out $y=\frac{TA-GD}{GZ-BT}=\frac{1}{10}\,l.$ or 2 Shillings, $e=\frac{A+By}{g}=\frac{3}{20}\,l.$ or $3\,s.$ $a=\frac{p-eg-yl}{b}=\frac{1}{4}\,l.$ or $5\,s.$

Problem XLIV.

590. To determine the *Proportion* of *two Ingredients*, whereof a *Mixture* is compoſed from the *Specific Gravity* (*i. e.* the Weight of the ſame Quantity) both of the *Compoſition* and the *Ingredients* given ſeparate.

Effection.

Let the Number of Cubic Inches, Pints, Quarts, &c. in the Compoſition be expreſſed by b, that of the one Ingredient by a, and of the other by e. Put

Put q for the Weight of one Cubic Inch, Pint, Quart, &c. of the Composition respectively, p for the Weight of the same Quantity of a, and r for the Weight of the like Quantity of e.

$$\text{Then}\left\{\begin{array}{l} 1:p=a:pa \text{ the Weight of } a \\ 1:q=b:qb \text{ the Weight of } b \\ 1:r=e:re \text{ the Weight of } e \end{array}\right\}$$

Consequently because

$$\left.\begin{array}{ll} 1 & a+e=b \\ 2 & pa+re=qb \end{array}\right\} \textit{By the Question.}$$

Whence 3 $a=b\times\frac{q-r}{p-r}$, or $p-r:b=q-r:a$ Theorem I.

And 4 $e=b\times\frac{p-q}{p-r}$, or $p-r:b=p-q:e$ Theorem II.

$\therefore$ 5 $q-r:p-q=a:e$ Theorem III.

Q. E. F.

Example. A Sea Captain had bought 12 Gallons of the choicest Brandy, the Weight of a Pint of which he had experienced was in Proportion to so much Water, as 9 to 10. But when he came to make use of his Brandy, he found that it bare Proportion to the same Quantity of Water as $9\frac{1}{11}$ to 10; he was therefore convinced that his Cabbin-Boy (to whose Trust it was committed) had adulterated it with Water. It is therefore required to find the Quantity of Water that had been put into it instead of so much Brandy.

Here is given $b=12$, $9=r$, $9\frac{1}{11}=q$, and $10=p$, therefore $a=b\times\frac{q-r}{p-r}=1$ Gallon of Water put in; and $e=b\times\frac{p-q}{p-r}=11$ the Gallons of Brandy remaining.

Scolium II.

591. The Weight peculiar to each Species of Matter, whereby it is distinguished from all others, is called their *Specific Gravity*, the Knowledge of which being a Thing of very great Use, I shall therefore here insert.

I. A

I. *A Table of the* Specific *Gravities of several Fluids.*

A Cubic Inch Paris *Measure.*	*In Summer.* Oz.	Dr.	Gr.	*In Winter.* Oz.	Dr.	Gr.
Of Mercury	7	1	66	7	2	14
Oil of Vitriol		7	59		7	71
Spirit of Vitriol		5	33		5	38
Spirit of Nitre		6	24		6	44
Spirit of Salt		5	49		5	55
Aqua Fortis		6	23		6	35
Vinegar.		5	15		5	21
Diftill'd Vinegar		5	11		5	15
Burgundy Wine		4	67		4	75
Spirit of Wine		4	32		4	42
Pale Ale		5	1		5	9
Brown Ale		5	2		5	7
Cows Milk		5	20		5	25
Goats Milk		5	24		5	28
Urine		5	14		5	19
Spirit of Urine		5	45		5	53
Oil of Tartar		7	27		7	43
Oil of Olives.		4	53	Is frozen		
Oil of Turpent.		4	39		4	46
Sea Water		6	12		6	18
River Water		5	10		5	13
Spring Water		5	11		5	14
Diftilled Water		5	08		5	11

II. *A Table of the* Specific *Gravities of several Solids.*

℔	
$71\frac{1}{2}$ of Mercury	Is equal in Magnitude to one Hundred Pound Weight of Gold.
$60\frac{1}{2}$ of Lead	
$54\frac{1}{2}$ of Silver	
$47\frac{1}{3}$ of Copper	
45 of Brafs	
42 of Iron	
39 of Tin	
$38\frac{1}{2}$ of fine Tin	
26 of Loadftone	
21 of Marble	
14 of Stone	
$12\frac{1}{2}$ of Sulphur	
5 of Wax	
$5\frac{1}{3}$ of Water	

Problem XLV.

592. Suppofe two Bodies A and B, at the Diftance b from each other, tend to a certain Place between them, and A begins to move before B the Time b, moving c (Inches, Feet, or Miles) in the Time d; whereas B moves f (Inches, Feet, or Miles) in g Time. It is required to find how far A will be gone before it meets B.

Effection.

For the Diftance moved by A put a, then will $b-a$ exprefs the Space moved by B.

	1	$c:d=a:\frac{da}{c}$ the Time *A* moves.
	2	$f:g=b-a:\frac{bg-ga}{f}$ the Time *B* moves.
	3	$\frac{bg-ga}{f}+h=\frac{da}{c}$ by the Question.
Whence	4	$a=\frac{cbg+cfh}{fd+cg}$ the Distance gone by *A*. Theorem I.
And	5	$b-a=\frac{bfd-cfh}{fd+cg}$ the Distance gone by *B*. Theorem II.

Ex. gr. If $b=59$ Miles, $c=7$ Miles, $f=8$ Miles; and $d=2$ Hours, $g=3$ Hours, $h=1$ Hour; then $a=c\times\frac{bg+fh}{fd+cg}=35$ Miles, $b-a=f\times\frac{bd-ch}{fd+cg}=24$ Miles.

Problem XLVI.

593. Suppose two Bodies *A* and *B*, at the Distance *b* (Inches, Feet or Miles) from each other; whereof *A* begins to move *h* Time before *B*, following *B* at the Rate of *c* Space in *d* Time, whilst *B* flees before at the Rate of *f* Space in *g* Time. It is required to know whether *B* will be overtaken by *A* at that Rate; and if it will, in what Distance moving.

Effection.

For the Distance moved by *A* put *a*, then will $a-b$ express the Distance moved by *B*.

	1	$c:d=a:\frac{da}{c}$ the Time *A* moves
	2	$f:g=a-b:\frac{ga-gb}{f}$ the Time *B* moves.
	3	$\frac{bg-gb}{f}+h=\frac{da}{c}$ by the Question.
Whence	4	$a=c\times\frac{bg-fh}{cg-fd}$ the Space gone by *A*. Theorem I.
And	5	$a-b=f\times\frac{bd-ch}{cg-fd}$ the Space gone by *B*. Theorem II.

Ex.

Ex. gr. If $b=59$, $c=7$, $f=8$, $d=2$, $g=3$, $h=1$, then $a=236\frac{1}{7}$, $a-b=177\frac{2}{3}$.

PROBLEM XLVII.

594. To divide a given Number b into *Extream* and *Mean Proportional Geometrical*, or in other Terms, to divide b into two such Parts a, e, that a, e, b may be in $\div\!\div$.

Effection.

	1	$a+e=b$	By the Question.
	2	$a : e = e : b$	
	3	$ab=ee$	
1,	4	$ee=bb-2ba+aa$	
3, 4,	5	$ab=bb-2ba+aa$	
$5+2ba-aa$	6	$3ba-aa=bb$	
Whence	7	$a=\frac{3}{2}b-\overline{\frac{5}{4}bb}^{\frac{1}{2}}=b\times\overline{\frac{3}{2}-\frac{1}{2}\times5^{\frac{1}{2}}}=\frac{1}{2}b\times\overline{3-5^{\frac{1}{2}}}$	
∴	8	$e=b-a=b\times\overline{\frac{1}{2}\times5^{\frac{1}{2}}-\frac{1}{2}}=\frac{1}{2}b\times\overline{5^{\frac{1}{2}}-1}$	
Substitute	9	$\frac{3}{2}-\frac{1}{2}\times5^{\frac{1}{2}}=0.381966$1 near $=z$.	
And	10	$\frac{1}{2}\times5^{\frac{1}{2}}-\frac{1}{2}=0.6180339$ near $=1-z$	
Then	11	$a=bz$	
And	12	$e=b-bz$	

Ex. gr. If $b=4$. $a=bz=1.527864$, $e=b-bz=2.4721356$. If $b=5$, $a=bz=1.9098305$, $b-bz=3.0901695$, &c.

PROBLEM XLVIII.

595. To raise Theorems for the Computation of *Simple Interest*.

Effection.

Let C represent any Sum of Money lent out for the Time Y, and for the Interest which such Sum gains in that Time put R; then will P represent any other Principle lent out for the Time T, whose Interest at the same Rate is I. (In. 581.) If then R be put for the Rate or Interest of 1 *l.* $=C$, for one Year $=Y$, it is plain from Theorem I. in that Place, that the Interest of any Principle P for the Time T, at the same Rate will $=PTR=I$. And if A be put for the Amount of any Principle P in the Time T at the Rate R *per l. per Ann.* then.

$1\div\overline{PT+1}$

				Given	Sought
	1	$PRT+P=A$	Theorem I.	P, R, T,	A,
$1\div\overline{PT+1}$	2	$P=\frac{A}{RT+1}$	Theorem II.	A, R; T,	P,
$1-P$	3	$PRT=A-P$			
$3\div PR$	4	$T=\frac{A-P}{PR}$	Theorem III.	A, P, R,	T,
$3\div TP$	5	$R=\frac{A-P}{PT}$	Theorem IV.	A, P; T,	R,

Queſtion 1. What will be the Amount A of 334 *l.* 10 *s.* $=P$, if forborn for 2 Years 234 Days, or 2.640656 Years, $=T$, at 5 *l. per Cent. per Annum*; or which is the ſame Thing at 0.05 *l.* or 1 *s.* $=R$ *per l. per Ann. Simple Intereſt?*

Anſwer. $A=PRT+P=378{,}6649716$ *l.* or 378 *l.* 13 *s.* 3 *d.* 2 *qrs.* (Theo. I.)

Queſtion 2. What is the preſent Worth P of 378 *l.* 13 *s.* $3\frac{1}{2}$ *d.* $=A$, due two Years 234 Days hence $=T$, abating 5 *l. per Cent. per Annum*; or ſuppoſing 0.05 *l.* $=R$ *per l. per Annum*, Simple Intereſt?

Anſwer. $P=\frac{A}{RT+1}=334$ *l.* 10 *s.* Theorem II.

Queſtion 3. In what Time T, will 334 *l.* 10 *s.* $=P$ amount to 378 *l.* 13 *s.* $3\frac{1}{2}$ *d.* $=A$, at 0.05 *l.* $=R$ *per l. per Annum*, Simple Intereſt?

Anſwer. $T=\frac{A-P}{PR}=$ 2 Years 234 Days. Theorem III.

Queſtion 4. It is required to determine at what Rate R, will 334 l. 10 s. $=P$, in 2 Years 234 Days $=T$, amount to 378 *l.* 13 *s.* $3\frac{1}{2}$ *d.* $=A$, Simple Intereſt?

Anſwer. $R=\frac{A-P}{PT}=0.05$ *l.* (or 5 *l. per Cent. per Ann.*) Theorem IV.

PROBLEM XLIX.

596. To raiſe Theorems for the Computation of *Compound Intereſt.*

Compound Intereſt is when by Reaſon of Non-Payment, the Intereſt due at every foregoing Payment, at equal Times, is made Part of the Principle of the following Payment: So that the Principle bears the ſame Proportion, or Ratio, to the Amount of the firſt Payment, that the Amount of the firſt Payment bears to the Amount of the ſecond Payment, and that the Amount of the ſecond Payment bears to the Amount of the third; the third to the fourh,

fourth, &c. Thus, if 1 *l.* amount the first Year to 1.05 *l.* or 1 *l.* 1 *s.* then by continuing unpaid it will amount the second Year to $1.05 \times 1.05 = 1.1025$ *l.* the third Year to $\overline{1.05}^{3} = 1.157625$ *l.* the fourth Year to $\overline{1.05}^{4} = 1.21550625$ *l.* the fifth Year to $\overline{1.05}^{5} = 1.2762815$6, &c. Consequently the Amount of 1 *l.* for half a Year will be $\overline{1.05}^{\frac{1}{2}} = 1.02469507$ *l.* for a Quarter $\overline{1.05}^{\frac{1}{4}} = 1.01227223$ *l.* for a Month or $\frac{1}{12}$ of a Year $\overline{1.05}^{\frac{1}{12}} = 1.00407412$ *l.* for a Day or $\frac{1}{365}$ of a Year $\overline{1.05}^{\frac{1}{365}} = 1.00013368$ &c. *l.* &c.

If R be put for the Amount of 1 *l.* with its Interest for 1 Year, *i. e.* $R = 1.05$, at 5 *l. per Cent. per Ann.* $R = 1.06$ *l.* at 6 *l. per Cent.* $R = 1.07$ *l.* at 7 *l. per Cent.* &c. then R^2 will equal the Amount of 1 *l.* at two Years End; R^3 at three Years End; $R^{\frac{1}{4}}$ at a Quarter, $R^{\frac{1}{12}}$ at a Month, $R^{\frac{1}{365}}$ at a Day, &c. and universally R^T at the Number of Payment T. And by the *Rule of Three*, as 1 *l.* is to the Amount of 1 *l.* for the Number of Payments T, so is any Principle P to its Amount for the same Time or Number of Payments, *i. e.* $1 : R^T = P : PR^T$. Whence

				Given	Sought
	1	$PR^T = A$	Theorem I.	$P,\ T,\ R,$	$A,$
$1 \div R^T$	2	$P = \dfrac{A}{R^T}$	Theorem II.	$A,\ R,\ T,$	$P,$
$1 \div P$	3	$R^T = \dfrac{A}{P}$			
$1\,w^T$	4	$R = \left(\dfrac{A}{P}\right)^{\frac{1}{T}}$	Theorem III.	$A,\ P,\ T,$	$R,$

Question 1. What will be the Amount A of 4000 *l.* $= P$, if forborn $4 = T$ Years, at the Amount of 1.05 *l.* R *per l. per Annum*, Compound Interest?

Answer. $A = PR^T = 4862.025$ *l.* $= 4862$ *l.* 00 *s.* 06 *d.* Theo. I.

Question 2. What is the present Worth P of 4862 *l.* 00 *s.* 06 *d.* $= A$, due $4 = T$ Years hence, abating at the Rate of 1.05 *l.* $= R$ for the Amount of 1 *l.* *per Ann.* Compound Interest?

Answer. $P = \dfrac{A}{R^T} = 4000$ *l.* Theo. II.

Question 3. It is required to determine at what Rate R will 4000 *l.* $= P$ in $4 = T$ Years amount to 4862.025 *l.* $= A$, Compound Interest?

Answer. $R = \left(\dfrac{A}{P}\right)^{\frac{1}{T}} = 1.05$ *l.* or 1 *l.* 1 *s.* Theo. III.

Question 4. In what Time T will 4000 *l.* $=P$ amount to 48662.025 *l.* $=A$, at the Amount of 1.05 *l.* $=R$ *per l. per Ann.* Compound Intereſt?

This is performed by dividing the Quotient $\frac{A}{P}$ by R, and that Quotient again by R, and that again by R, *&c.* till the laſt Quotient be Unity, and the Number of ſuch Diviſions (which are four) will give T, *i. e.* $T=4$.

Problem L.

597. What three Numbers are thoſe a, e, y, whereof the Square of the firſt, added to the Product of the ſecond and third, equals l, the Square of the ſecond added to the Product of the firſt and third equals m, and the Square of the third added to the Product of the firſt and ſecond equals n. *Ex. gr.* Let $l=16$, $m=17$, $n=18$. See *Wallis*'s *Algebra*, p. 225.

Effection.

	1	$\left.\begin{matrix} aa+ey=l \\ ee+ay=m \\ yy+ae=n \end{matrix}\right\}$ *By the Queſtion.*
	2	
	3	
1—aa	4	$ey=l-aa$
4÷y	5	$e=\frac{l-aa}{y}$
3—yy	6	$ae=n-yy$
6÷a	7	$e=\frac{n-yy}{a}$
5, 7	8	$\frac{l-aa}{y}=\frac{n-yy}{a}$
8×ay	9	$la-a^3=ny-y^3$ or $y^3-ny-a^3+la=0$
Subſtit.	10	$A=1$, $B=0$, $C=-n$, $D=la-a^3$
Then	11	$Ay^3+By^2+Cy+D=0$
5×7	12	$ee=\frac{ln-lyy-na^2+a^2y^2}{ay}$
2—ay	13	$ee=m-ay$
12, 13	14	$\frac{ln-lyy-na^2+a^2y^2}{ay}=m-ay$
Whence	15	$\begin{matrix}+2a^2\\-l\end{matrix}\, yy-may\, \begin{matrix}-naa\\+ln\end{matrix}=0$
Subſtit.	16	$2a^2-l=F$, $G=-ma$, $H=ln-naa$
Then	17	$Fyy+Gy+H=0$.

There-

Therefore restoring the Values of *A*, *B*, *C*, *D*, *F*, *G*, *H* in the 10th and 16th Steps, and (by In. 530. Rule II.) we have this Equation with y exterminated.

	18	$8a^{12}=28a^{10}\begin{array}{l}+38ll\\-2mn\end{array}a^{8}\begin{array}{l}-25lll\\+7lmn\\-mmm\\-nnn\end{array}a^{6}\begin{array}{l}+8l^4\\-7l^2mn\\+lm^3\\+ln^3\\+m^2n^2\end{array}a^{4}\begin{array}{l}-l^5\\+2l^3mn\\-lm^2n^2\end{array}a^{2}=0$
$18\div\overline{8a^2}$	19	$a^{10}-\frac{7}{2}a^{8}\begin{array}{l}+\frac{19}{4}ll\\-\frac{1}{4}mn\end{array}a^{6}\begin{array}{l}-\frac{25}{8}l^3\\+\frac{7}{8}lmn\\-\frac{1}{8}m^3\\-\frac{1}{8}n^3\end{array}a^{4}\begin{array}{l}+l^4\\-\frac{7}{8}lmn\\+\frac{1}{8}lm^3\\+\frac{1}{8}ln^3\\+\frac{1}{8}m^2n^2\end{array}a^{2}\begin{array}{l}-\frac{1}{8}l^5\\+\frac{1}{4}l^3mn\\-\frac{1}{8}lm^2n^2\end{array}=0$
Make	20	$u=2a^2$ then (by In. 486.)
$aa\times\overline{2}$	21	$u^5-7lu^4\begin{array}{l}+19ll\\-mn\end{array}u^{3}\begin{array}{l}-25l^3\\+7lmn\\-m^3\\-n^3\end{array}u^{2}\begin{array}{l}+16llll\\-14l^2mn\\+2lm^3\\+2ln^3\\+2m^2n^2\end{array}u\begin{array}{l}-4l^5\\+8l^3mn\\-4lm^2n^2\end{array}=0$
	22	$u-2l=0$ or $u=2l=32=2aa$ (In. 424.)
$22\div2$	23	$aa=16$
	24	$a=4$. But by this; $\frac{l-aa}{y}=e=0$ in the 5th Step, therefore another Value of a must be found by dividing the Equation in the 21st Step, by $u-2l=0$, and so reducing the Equation of five Dimensions into one of four.
	25	$u^4-5lu^3\begin{array}{l}+9ll\\-mn\end{array}u^{2}\begin{array}{l}-7lll\\+5lmn\\-m^3\\-n^3\end{array}u\begin{array}{l}+2l^4\\-4llmn\\+2m^2n^2\end{array}=0$
Or in Numbers	26	$u^4-80u^3+1998u^2-14937u+5000=0$
Whence	27	$u=2aa=12.75644179448074402$, &c. (In. 509. *Ex.* 2.)
$27\div2$	28	$aa=$ 6.37822089724037 2 &c.
$28\,w^2$	29	$a=$ 2.525513986744158 near
9.	30	$24.2999377013 82096=18y-y^3$
Whence	31	$y=3.2405806816 17174$
5	32	$e=\frac{l-aa}{y}=2.9691527686 19848$

Proof

Proof.

$$aa = 6.3782208972403772$$
$$ey = 9.621779102759628$$

$$aa+ey = 16.000000000000000$$

$$ee = 8.815868163402909$$
$$ay = 8.184131836597093$$

$$ee+ay = 17.000000000000002$$

$$yy = 10.501363154070430$$
$$ay = 7.498636845929567$$

$$yy+ae = 17.9999999999999997 = 18 \text{ near.}$$

Scolium III.

598. Dr. *Wallis* gives three other Anſwers to the foregoing Queſtion in Numbers according to the three remaining Roots of the Equation in the 25th Step. Thus,

The ſecond Value of u is 0.350987046 near, according to which

$$\left.\begin{array}{l} a=0.4189194707 \\ e=3.9122268666 \\ y=4.0448846707 \end{array}\right\}\text{near.}$$

The third Value of u is 34.83228028, according to which

$$\left.\begin{array}{l} a=+4.173264926 \\ e=+4.287022553 \\ y=-0.330331815 \end{array}\right\}\text{near.}$$

The fourth Value of u is 32.06029088, according to which

$$\left.\begin{array}{l} a=+4.0037664077 \\ e=-0.007099744 \\ y=+0.2459893 \end{array}\right\}\text{near.}$$

CHAP.

CHAP II.

Of Arithmetical and Geometrical Progression.

Problem LI.

599. FROM any three given, of these five, d, g, l, n, s, to find the rest (In. 206.)

Effection.

				Given	Sought
	1	$g=l+nd-d$ (In. 207.)	Theo. I.	$d, l, n,$	$g.$
$1+d$	2	$g+d=l+nd$			
$2-nd$	3	$g+d-nd=l$	Theo. II.	$d, g, n,$	$l.$
$2-l$	4	$g+d-l=nd$			
$4\div d$	5	$\frac{g+d-l}{d}=n$	Theorem III.	$d, g, l,$	$n.$
$1-l$	6	$g-l=nd-d$			
$6\div\overline{n-1}$	7	$\frac{g-l}{n-1}=d$	Theo. IV.	$g, l, n,$	$d.$
	8	$\frac{nl+ng}{2}=n\times\frac{l+ng}{2}=s$ (In. 212)	Theo. V.	$g, l, n,$	$s.$
8×2	9	$nl+ng=2s$			
$9\div\overline{l+g}$	10	$n=\frac{2s}{l+g}$	Theo. VI.	$g, l, s,$	$n.$
$9-ng$	11	$nl=2s-ng$			
$11\div n$	12	$l=\frac{2s-ng}{n}$	Theo. VII.	$g, n, s,$	$l.$
$9-nl$	13	$ng=2s-nl$			
$13\div n$	14	$g=\frac{2s-nl}{2}$	Theo. VIII.	$l, n, s,$	$g.$
$1=14$	15	$\frac{2s-nl}{n}=l+nd-d$			
$15\times n$	16	$2s-nl=nl+n^2d-nd$			
$16+nl$	17	$2s=2nl+nnd-nd$			
$17\div2$	18	$s=\frac{2nl+nnd-nd}{2}=nl+nd\times\frac{n-1}{2}$	Theo. IX.	$d, l, n.$	$s.$

			Given.	Sought
$17-2nl$	19	$2s-2nl=n^2d-nd$		
$19\div\overline{nn-n}$	20	$\frac{2s-2nl}{n^2-n}=d$ Theo. X.	l, n, s,	d.
$17-n^2d+nd$	21	$2s-n^2d+nd=2nl$		
$21\div 2n$	22	$\frac{2s-n^2d+nd}{2n}=l=\frac{s}{n}-d\times\frac{\overline{n-1}}{2}$ Theo. XI.	d, n, s,	l.
$17\div d$	23	$\frac{2s}{d}=n^2-n+\frac{2l}{d}n$		
Whence	24	$\frac{\overline{8ds+dd+4ll-4dl}\vert^{\frac{1}{2}}+d-2l}{2d}=n$ Theo. XII.	d, l, s,	n.
$3=12$	25	$g+d-nd=\frac{2s-ng}{n}$		
$25\times n$	26	$ng+nd-n^2d=2s-ng$		
$26+ng$	27	$2ng+nd-n^2d=2s$		
$27\div 2$	28	$\frac{2ng+nd-n^2d}{2}=ng+nd\times\frac{1-n}{2}=s$ Th. XIII.	d, g, n,	s.
$27-nd+n^2d$	29	$2ng=2s-nd+n^2d$.		
$29\div 2n$	30	$g=\frac{2s-nd+n^2d}{2n}=\frac{s}{n}-d\times\frac{\overline{1-n}}{2}$ Th. XIV.	d, n, s,	g.
$29-s$	31	$2ng-2s=n^2d-nd$		
	32	$\frac{2ng-2s}{n^2-n}=d$ Theo XV.	g, n, s,	d.
$29+nd-n^2d$	33	$2ng+nd-n^2d=2s$		
$33\div d$	34	$\frac{2g}{d}n+n-n^2=\frac{2s}{d}$		
Whence	35	$n=\frac{2g+d-\vert\overline{4gg+dd+4dg-8ds}\vert^{\frac{1}{2}}}{2d}$ Theo. (XVI.	d, g, s,	n.
$5=10$	36	$\frac{2s}{l+g}=\frac{g+d-l}{d}$		
$36\times d$	37	$\frac{2ds}{l+g}=g+d-l$		
$37\times\overline{l+g}$	38	$2ds=ld-ll+g^2+gd$		
$38\div 2d$	39	$s=\frac{ld-l^2+g^2+gd}{2d}$ Theo. XVII.	d, g, l,	s.
$38-ld-gd$	40	$2sd-ld-gd=g^2-l^2$		

$40\div\overline{2s-l-g}$

				Given	Sought
$40 \div \overline{2s-l-g}$	41	$d=\dfrac{g^2-l^2}{2s-l-g}$	Theo. XVIII.	$g, l, s,$	$d.$
$40+gd$	42	$2sd-ld=g^2-l^2+gd$			
$40+ll$	43	$2sd-ld+l^2=g^2+dg$			
Whence	44	$\overline{2sd-ld+l^2+\frac{1}{4}dd}^{\frac{1}{2}}-\frac{1}{2}d=g.$	Theo. XIX.	$d, l, s,$	$g.$
$43-2sd$	45	$l^2-ld=g^2+dg-2sd$			
Whence	46	$l=\overline{g^2+dg-2sd+\frac{1}{4}dd}^{\frac{1}{2}}+\frac{1}{2}d$ Or if d be $>l$ $l=\frac{1}{4}d-\overline{2sd+\frac{1}{4}dd-g^2-dg}^{\frac{1}{2}}$	Theo. XX.	$d, g, s,$	$l.$
	&c.			&c.	

SCOLIUM IV.

600. The foregoing Theorems may be illustrated by the following Examples.

Question 1. A certain Man had 13 Children, the youngest of which was 3 Years old, and every one of the rest was 2 Years and a Quarter (or 2.25 Years) elder than the other: It is required to know what was the Age of the eldest; and also the Sum of all their Ages.

Here is given $l=3$, $n=13$, and $d=2\frac{1}{4}$, to find $g=$ the Age of the Eldest, and $s=$ the Sum of all their Ages.

Answer. $g=l+nd-d=30$, *per* Theo. I. and $s=\dfrac{2nl+nnd-nd}{2}=214\frac{1}{2}$ *per* Theo. IX.

Question 2. Suppose a Person had 13 Inclosures of Ground, each of which was $2\frac{1}{4}$ Acres more than another, and the largest contained 30 Acres: It is required to determine how many Acres were contained in the smallest, and also what Quantity of Ground he had in all.

Here is given $n=13$, $d=2\frac{1}{4}$, and $g=30$, to find $l=$ the Acres contained in the least Inclosure, and $s=$ the Sum of all the Acres he had.

Answer. $l=g+d-nd=3$, *per* Theo. II. and $s=\dfrac{2ng+nd-n^2d}{2}=314.5$, *per* Theorem XIII.

Question 3. One had a certain Number of Horses, each of which he valued at 2 *l.* 5 *s.* or $2\frac{1}{4}$ *l.* above the other; the best he prized at 30 *l.* and the worst at 3 *l.* It is required to find the Number of Horses he had, and what was the Price of them all taken together.

Here

Here is given $d=2\frac{1}{4}$, $g=30$, and $l=3$, to find $n=$ the Number of Horses, and $s=$ the Sum of all their Prizes.

Answer. $n=\frac{g+d-l}{d}=13$, *per* Theorem III. and

$$s=\frac{ld-l^2+g^2+gd}{2d}=214.5. \text{ per Theorem XVII.}$$

Question 4. Suppose 13 Eggs placed in a direct Line from a Basket, at equal Distances from each other; the nearest of which is 3 Yards from the Basket, and the furthest off 30 Yards from it: It is required to find their Distance from each other; and also how many Yards that Man must run who is to gather up those Eggs one by one, beginning with the nearest, and still returning with every Egg to the Basket.

Here is given $n=13$, $l=2\times3$ or 6, and $g=2\times30$, or 60 (by reason of the Man's going and returning with every Egg,) to find $d=$ the Space which the Man runs for every following Egg more than he did for the foregoing one, the Half of which is the Distance between Egg and Egg; and $s=$ the Number of Yards which he runs in gathering up all.

Answer. $d=\frac{g-l}{n-1}=4\frac{1}{2}$, *per* Theo. IV.
$s=\frac{nl+ng}{2}=429$, *per* Theo. V.
$\therefore$ The Distance between every Egg is $2\frac{1}{4}$.

Question 5. A Recruiting Officer went to a certain Place, where he tarried a certain Number of Days, the first Day he listed 9 Men, and every following Day he encreased by a certain Number more than he listed the foregoing one, till the last Day he listed 30, and had in all 156 fresh Men: It is required to find the Number of Days he tarried, and how many Men he encreased each following Day more than he had the foregoing one.

Here is given $l=9$, $g=30$, and $s=156$, to find $n=$ the Number of Days he tarried, and $d=$ the Number of Men he listed on each following Day more than he did the foregoing one.

Answer. $n=\frac{2s}{l+g}=8$, *per* Theorem VI.

$$d=\frac{g^2-l^2}{2s-l-g}=3, \text{ per Theorem XVIII.}$$

Question 6. A certain Man had $n=13$ Children, who differed all alike in their Ages, the eldest was $g=30$ Years old; and the Sum of all their Ages, was

was $s=214\frac{1}{2}$ Years. It is required to find l the Age of the youngest, and alſo d the Difference of their Ages.

Anſwer. $l=\dfrac{2s-ng}{n}=3$, *per* Theorem VII.

$d=\dfrac{2ng-2s}{n^2-n}=2\frac{1}{4}$, *per* Theorem XV.

Queſtion 7. Suppoſe a Perſon had $n=13$ Incloſures of Ground, each the ſame Quantity of Acres bigger than the other, the leaſt of which contained $l=3$ Acres, and the Sum of the Acres contained in them all was $s=214\frac{1}{2}$ Acres: It is required to find $g=$ the Number of Acres contained in the greateſt, $d=$ the Number of Acres by which every next greater exceeds the next leſſer.

Anſwer. $g=\dfrac{2s-nl}{n}=30$, *per* Theorem VIII.

$d=\dfrac{2s-2nl}{n^2-n}=2\frac{1}{4}$, *per* Theorem X.

Queſtion 8. One had $n=13$ Horſes, each of which he valued at $d=2\frac{1}{4}\,l.$ more than the other, and the Sum of the Prices of all was $s=214\,l.\ 10\,s.$ It is required to determine the Prizes of the worſt l and the beſt g ſeparately; and conſequently the Prices of each of the reſt.

Anſwer. $l=\dfrac{s}{n}-\dfrac{nd-d}{2}=3$, *per* Theorem XI.

$g=\dfrac{s}{n}=\dfrac{d-nd}{2}=30$, *per* Theorem XIV.

Queſtion 9. Suppoſe a Number of Eggs were placed in a direct Line from a Baſket at $\frac{1}{2}d=2\frac{1}{4}$ Yards from each other, the neareſt of which is $\frac{1}{2}l=3$ Yards from the Baſket, and it would require a Perſon to walk $s=429$ Yards to take them up ſingly, and return with each to the Baſket. It is required to determine $n=$ the Number of Eggs, and $g=$ the Number of Yards which the fartheſt lies from the Baſket.

Anſwer. $n=\dfrac{\overline{8ds+dd+4ll-4dl}^{\frac{1}{2}}+d-2l}{2d}=13$, *per* Theorem XII.

$g=\overline{2ds-ld+ll+\frac{1}{4}dd}^{\frac{1}{2}}-\frac{1}{2}d=30$, *per* Theorem XIX.

Queſtion 10. A Recruting Officer went to a certain Place, where he tarried a certain Number of Days, and every Day encreaſed his Company $d=3$ Men more than he encreaſed it the foregoing one, and the laſt Day he en-

N creaſed

creafed it $g=30$ Men, gaining in all $s=156$ fresh Men. It is required to determine $n=$ the Number of Days he tarried, and how many Men l he listed the first Day, and consequently every other Day.

$$\textit{Anfwer.}\ n=\frac{2g+d-\overline{4gg+dd+4dg-8ds}^{\frac{1}{2}}}{2d}=8, \textit{per}\ \text{Theorem XVI.}$$

$$l=\overline{gg+dg+\tfrac{1}{4}dd-2sd}^{\frac{1}{2}}+\tfrac{1}{2}d=9, \textit{per}\ \text{Theorem XX.}$$

Corollary II.

601. From hence is learned to compute the *Amount* of any yearly Rent or *Penfion*, if forborn for any Number of Years, according to any propofed *Rate*, at *Simple Intereft*. For if

$U=$ the *Penfion* yearly, half yearly, or quarterly, &c. Rent.
$T=$ the *Time* of its Forbearance, or the Number of the Payments forborn.
$R=$the *Rate* or *Intereft* of 1 *l.* for 1 Year, as (In. 595.)
$A=$ the *Amount* of the Penfion with its Intereft for the propofed Time.

Then will

$U=$ the *Amount* of the firft Year, or Half Year, &c. without Intereft.
$2U+$ the Intereft $1U=$ the *Amount* of the *fecond* Year, or Half Year, &c.
$3U+$ the Intereft of $1U+2U=$ the *Amount* of the *third* Year.
$4U+$ the Intereft of $1U+2U+3U=$ the *Amount* of the *fourth* Year, &c.

And univerfally

$TU+$ the Intereft of $1U+2U+3U$ &c. to $+TU-U$ will equal the Amount of the T Number of Years, Half Years, Quarters, &c. according to the Term of the Penfion U.

But $RU=$ the Intereft of $1U$.
$2RU=$ the Intereft of $2U$.
$3RU=$ the Intereft of $3U$. And univerfally
$TRU=$ the Intereft of TU. *i. e.* $1:R=TU:RTU$. And $1:R=\overline{T-1}\times U:\overline{T-1}\times RU$. Therefore $A=TU+RU+2RU+3RU+4RU$, &c. to $\overline{T-1}\times RU$. *i. e.* A equals $TU+$ the Sum of a Series of Terms in $\div$ whofe firft Term and common Difference is RU, Number of Terms $T-1$, and laft Term $\overline{T-1}\times RU$. But fuch a Sum is equal to $\dfrac{TTRU-TRU}{2}$ *per* Theorem Vth, IXth, XIIIth, and XVIIth. (In. 600.) Therefore

$A=$

				Given	Sought
	1	$A=TU+\frac{TTRU-TRU}{2}$ or			
		$A=TU+TRU\times\overline{\frac{T-1}{2}}$ Theo. I.		$R, T, U,$	A
1×2	2	$2A=2TU+TTRU-TRU$			
$2\div\overline{2T+TTR-TR}$	3	$\frac{2A}{2T+TTR-TR}=U$ Theo. II.		$A, R, T,$	$U,$
$2\div RU$	4	$\frac{2A}{RU}=TT+\frac{2T}{R}-T$			
Whence	5	$\frac{\overline{\frac{2AR}{U}+1+\frac{1}{4}RR-R}^{\frac{1}{2}}+\frac{1}{2}R-1}{R}=T$	Th. (III.	$A, R, U,$	$T,$
$2-2TU$	6	$2A-2TU=TTRU-TRU$			
$6\div\overline{TTU-TU}$	7	$\frac{2A-2TU}{TTU-TU}=R$ Theo. IV.		$A, T, U,$	$R,$

Ex. gr. Question 1. What is the Amount $=A$ of 500 *l.* $=U$ yearly Rent, forborn $9=T$ Years, at 0.05 *l.* $=R$ *per* Pound *per Annum* Simple Intereſt?

Anſwer. $A=TU+TRU\times\overline{\frac{T-1}{2}}=5400$ *l. per* Theorem I.

Queſtion 2. What is the Amount $=A$ of 125 *l.* $=U$ quarterly Rent, forborn 36 Quarters $=T$, at 0.0125 *l.* $=R$ *per* Pound *per* Quarter, Simple Intereſt?

Anſwer. $A=TU+TRU\times\overline{\frac{T-1}{2}}=5484$ *l.* 7 *s.* 06 *d.*

Queſtion 3. What quarterly Rent $=U$ being forborn 9 Years, or 36 Payments$=T$, will amount to 5484.375 *l.* $=A$ (or 5484 *l.* 7 *s.* 6 *d.*) allowing 0,0125 *l.* $=R$ *per* Pound *per* Quarter for each Payment as it becomes due: Simple Intereſt?

Anſwer. $U=\frac{2A}{2T+TTR-TR}=125$ *l. per* Theorem II.

Queſtion 4. In how many Payments $=T$ will 125 *l.* quarterly Rent $=U$ amount to 5484.375 *l.* $=A$ allowing 0.0125 *l.* $=R$ *per* Pound *per* Quarter for the Forbearance of the Payments as they become due?

Anſwer.

Answer. $T=\dfrac{\overline{\dfrac{2AR}{U}+1+\frac{1}{4}RR-R}\,\Big|^{\frac{1}{2}}+\frac{1}{2}R-1}{R}=36$, *per* Theorem III.

Question 5. If 125 *l.* quarterly Rent $=U$, forborn for 36 Payments $=T$, amount to 5484.375 *l.* $=A$, allowing Simple Intereſt for every Payment as it becomes due; what is the Rate of Intereſt *per* Pound per Quarter $=R$?

Answer. $R=\dfrac{2A-2TU}{TTU-TU}=0.0125$ *l.* or 03 *d.* *per* Theorem IV.

Corollary III.

602. And hence alſo is learned to compute the preſent Value or Price $=P$ of any yearly Rent or Penſion at Simple Intereſt for any propoſed Time or Number of Payments to come. For, repreſenting all Things as before, then the Amount $=A$ of any Penſion, yearly, half yearly, quarterly, *&c.* Rent $=U$, for any propoſed Time or Number of Payments forborn $=T$, will be $TU+TRU\times\dfrac{T-1}{2}$ (In. 601.) And again, the Amount $=A$ of any Principle P, if forborn for any Time $=T$ at any given Rate of Simple Intereſt $=R$ be $PRT+P$. (In. 595.) *i. e.* $A=TU+TRU\times\dfrac{T-1}{2}$ or $\dfrac{2TU+TTRU-TRU}{2}$, and $A=PRT+P$, conſequently

			Given	Sought
	1	$PRT+P=\dfrac{2TU+TTRU-TRU}{2}$ (In. 21.)		
$1\div\overline{RT+1}$	2	$P=\dfrac{2TU+TTRU-TRU}{2RT+2}$ Theo. I.	$R,T,U,$	$P,$
1×2	3	$2PRT+2P=2TU+TTRU-TRU$		
$3\div\overline{2T+TTR-TR}$	4	$\dfrac{2PRT+2P}{2T+TTR-TR}=U$ Theo. II.	$P,R,T,$	$U,$
$3-2PRT$	5	$2P=2TU+TTRU-TRU-2PRT$		
$5-2TU$	6	$2P-2TU=TTRU-TRU-2PRT$		
$6\div\overline{TTU-TU-2PT}$	7	$\dfrac{2P-2TU}{TTU-TU-2PT}=R$ Theo. III.	$P,T,U,$	$R,$
$5\div RU$	8	$\dfrac{2P}{RU}=TT-T-\dfrac{2PT}{U}+\dfrac{2T}{R}$		

Subſtitute

			Given	Sought
Substitute	9	$\frac{2}{R}-\frac{2P}{U}-1=x$		
8, 9.	10	$\frac{2P}{RU}=TT\pm Tx$		
Whence	11	$\overline{\frac{2P}{RU}+\frac{xx}{4}}\Big\vert^{\frac{1}{2}}\pm\frac{1}{2}x=T$. Theo. IV.	$P,R,U,$	T.

Question 1. What is the present Worth P of 500 *l.* $=U$ yearly Rent at 0.05 *l.* $=R$ *per* Pound *per Annum*, to continue $9=T$ Years?

Answer. $P=TU\times\frac{2+R\times\overline{T-1}}{2RT+2}=3724.137931$ *l.* or 3724 *l.* 02 *s.* 09 *d.* *per* Theorem I.

Question 2. What yearly Rent $=U$ may be purchased for 3724.138 *l.* $=P$, to continue the Term of $9=T$ Years, allowing 0.05 *l.* $=R$ *per* Pound *per Annum* Interest?

Answer. $U=2P\times\frac{TR+1}{2T+TTR-TR}=500$ *l.* *per* Theorem II.

Question 3. For what Time T may a Pension of 500 *l.* $=U$ *per Annum* be enjoyed for 3734.138 *l.* $=P$, ready Money, at 0.05 *l.* $=R$ *per* Pound *per Annum* Simple Interest, for the Time every Payment is made before it becomes due?

Answer. $T=\overline{\frac{2P}{RU}+\frac{xx}{4}}\Big\vert^{\frac{1}{2}}-\frac{x}{2}=9$ Years.

Question 4. What is the present Worth $=P$ of 500 $l=U$ yearly Rent, at 0.05 *l.* $=R$ *per* Pound *per Annum* Simple Interest, to continue for $9=T$ Years to come; the first Year of which is not to commence till $5=t$ Years after the Payment is made.

Here find $P=TU\times\frac{2+R\times\overline{T-1}}{2RT+2}$ (*per* Theo. I.) $=3724.137931$ *l.* the present Worth of the Annuity, supposing it to be immediately entered upon, as before: But in regard it is not to be entered upon till $5=t$ Years hence, therefore the last Sum found, *viz.* 3724.137931 *l.* is to looked upon as the Amount of P for $5=t$ Years, at 0.05 *l.* *per* Pound *per Annum*, *i. e.* $3724.1379=A=PRt+P$ (In. 595.) Whence $P=\frac{A}{tR+1}$ or (substituting $A=PRt+$

 $P=$

$P = \frac{2TU + TTRU - TRU}{2RT + 2}$) $P = \frac{2TU + TTRU - TRU}{2RRTt + 2RT + 2Rt + 2} = 2979.3103\,l.$ or 2979 *l.* 06 *s.* $02\frac{1}{2}$ *d.* near, for the Price required. From which laſt Theorem, if need be, may alſo be raiſed Theorems for finding *T*, *U*, and *R* as above.

Problem LII.

603. From any three given, of theſe four *l*, *g*, *r*, *s*, (In. 218.) to find the fourth.

Effection.

				Given	Sought
	1	$sr - gr = s - l$ (In. 225.)			
$1 \div \overline{s-g}$	2	$r = \frac{s-l}{s-g}$	Theo. I.	*g*, *l*, *s*,	*r*,
$1 + gr$	3	$sr = s - l + gr$			
$3 - s$	4	$sr - s = gr - l$			
$4 \div \overline{r-1}$	5	$s = \frac{gr-l}{r-1}$	Theo. II.	*g*, *l*, *r*,	*s*,
$4 + l$	6	$sr - s + l = gr.$			
$6 \div r$	7	$s - \frac{s+l}{r} = g$	Theo. III.	*l*, *r*, *s*,	*g*,
$6 - \overline{sr - s}$	8	$l = gr + s - sr$	Theo. IV.	*g*, *r*, *s*,	*l*.

Scholium V.

604. The foregoing Theorems may be illuſtrated by the following Examples.

Queſtion 1. Suppoſe a Grain of Wheat and its Product were ſown for a certain Number of Years, every Grain having always the ſame Increaſe, ſo that the Increaſe of the laſt Year is 10000000 Grains, and the Sum of every Year's Increaſe with the firſt Grain is 11111111 Grains. It is required to find at what Rate it multiplied every Year, and how many Years it was ſown.

Here is given $1 = l$, $10000000 = g$, and $11111111 = s$ to find $r =$ the Rate required, and $n - 1 =$ the Number of Years it was ſown.

Anſwer. $r = \frac{s-l}{s-g} = 10$, Theo. 1. which ſhews that it increaſed yearly in a tenfold Proportion.

Then becauſe $g = lr^{n-1}$ (In. 219.) and conſequently $\frac{g}{l} = r^{n-1}$ therefore divide $\frac{g}{l} = 10000000$ by $r = 10$, and that Quotient again by $r = 10$, &c. and that Quotient $= 100000$ again by $r = 10$, &c. continuing ſo doing till

till the laſt Quotient be Unity, which will be done at ſeven Diviſions, *i. e.* $\frac{g}{lr^7}=1$ or $\frac{g}{l}=r^7=r^{n-1}$, whence $7=n-1$ the Number of Years the Corn had to increaſe in.

Queſtion 2. Suppoſe the Nails in a Horſe's Shoes to be 28, *viz.* 7 for each Foot; and the Horſe is valued at a Farthing the firſt Nail, two Farthings the ſecond, four the third, eight the fourth, *&c.* every following Nail doubling the Price of the foregoing one, ſo long as any remains. It is required to find the Horſe's Price on that Condition,

Here is given $l=1$, $r=2$, $n=28$, and conſequently $g=lr^{n-1}$ or $lr^{27}=r^5 \times r^5 \times r^{10} \times r^7=134217728$, to find s, Theo. II. Thus, $s=\frac{gr-l}{r-1}=268435455$ Farthings, or 279620 *l.* 5 *s.* 3¾ *d.* 3 *qrs.* the Price of the Horſe required.

Corollary IV.

605. Hence is learned the Method of computing the Amount A of any Penſion or Annuity U, if forborn for any Number of Years or Times of Payment T, according to any propoſed Rate, at Compound Intereſt. For if $R=$ the Amount of 1 *l.* (In. 596.) and $U=$ the firſt Payment, or firſt Year's Rent without Intereſt; then will

$U+RU=\left\{\begin{array}{l}\text{The ſecond Year's Rent} + \text{the Amount of the firſt Year's Rent}\\ \text{with its Intereſt.}\end{array}\right.$

$U+RU+R^2U=\left\{\begin{array}{l}\text{The third Year's Rent} + \text{the Amounts of the firſt and}\\ \text{ſecond Years Rents.}\end{array}\right.$

$U+RU+R^2U+R^3U=\left\{\begin{array}{l}\text{The fourth Year's Rent} + \text{the Amounts of the}\\ \text{firſt, ſecond and third Year's Rents.}\end{array}\right.$

$U+RU+R^2U+R^3U+R^4U=\left\{\begin{array}{l}\text{The fifth Year's Rent with the Amounts}\\ \text{of the firſt, ſecond, third and fourth}\\ \text{Year's Rents.}\end{array}\right.$

And univerſally

$U+RU+R^2U+R^3U, \&c. \text{ to } R^{T-1}U=\left\{\begin{array}{l}\text{The } T \text{ Number of Payments, with}\\ \text{the Amounts of the firſt, ſecond,}\\ \text{third, \&c. to } \overline{T-1} \text{ Number of}\\ \text{Payments.}\end{array}\right.$

Therefore $A=U+RU+R^2U+R^3U+R^4U$, *&c.* to $R^{T-1}U$, *i. e.* A equals the Sum of a Series of Terms in $\div\div$, whoſe firſt Term is U, common Multiplier R, Number of Terms T, and laſt Term is $R^{T-1}U$: But ſuch a Sum is equal to $U \times \frac{R^T-1}{R-1}$ (In. 603. Theo. II.)

Whence

			Given	Sought
Whence	1	$A=\frac{R^TU-U}{R-1}$ Theo. I.	$R, T, U,$	$A,$
$1\times\overline{R-1}$	2	$RA-A=R^TU-U$		
$2\div\overline{R^T-1}$	3	$\frac{RA-A}{R^T-1}=U$ Theo. II.	$A, R, T,$	$U,$
$2-R^TU+A$	4	$RA-R^TU=A-U$		
$4\div U$	5	$\frac{A}{U}R-R^T=\frac{A-U}{U}$ Theo. III.	$A, U, T,$	$R,$
$2+U$	6	$RA-A+U=R^TU$		
$6\div U$	7	$\frac{RA-A+U}{U}=R^T$ Theo. IV.	$A, R, U,$	$T.$

Ex. gr. Question 1. What is the Amount $=A$ of 500 *l.* $=U$ yearly Rent, forborn $9=T$ Years, at the Amount of 1.05 *l.* $=R$ *per* Pound *per Annum*, Compound Intereſt?

Anſwer. $A=\frac{R^TU-U}{R-1}=5513.2822$ *l.* $=5513$ *l.* 05 *s.* 07 *d.* 03 *qrs.* near.

Queſtion 2. What yearly Rent $=U$ being forborn $9=T$ Years, will amount to 5513.2822 *l.* $=A$, at the Amount of 1.05 *l.* $=R$ *per* Pound *per Annum*, Compound Intereſt?

Anſwer. $U=\frac{RA-A}{R^T-1}=500$ *l.* *per* Theo. II.

Queſtion 3. If 500 *l.* yearly Rent $=U$, forborn for $9=T$ Years amount to 5513.2822 *l.* $=A$. What is the Amount $=R$ *per* Pound *per Annum?*

Anſwer. $\frac{A}{U}R-R^T=\frac{A-U}{U}$ Theo. III. According to which R will be found $=1.05$ *l.*

Queſtion 4. In how many Payments $=T$ will 500 *l.* $=U$ yearly Rent, amount to 5513.2822 *l.* $=A$, at the Amount of 1.05 *l.* $=R$ *per* Pound *per Annum?*

Anſwer. $R^T=\frac{RA-A+U}{U}=1.55132822$ *l.* Theo. IV.

Then 1.55132822 being divided by $R=1.05$ and that Quotient by $R=1.05$, and that Quotient again by $R=1.05$, &c. till the laſt Quotient $=1$, the Number of Diviſions will be $9=T$ required.

606. And

COROLLARY V.

606. And hence also is learned to compute the present Value P of any Pension, yearly, half yearly, or quarterly, *&c.* Rent $=U$, at compound Interest for any proposed Time or Number of Payments to come T. For $A=\frac{R^TU-U}{R-1}$ (In. 605.) and $A=PR^T$ (In. 432.) Consequently

			Given	Sought
	1	$PR^T=\frac{R^TU-U}{R-1}$		
$1\div R^T$	2	$P=\frac{R^TU-U}{R^{T+1}-R^T}=U\times\frac{1-\frac{1}{R^T}}{R-1}$ Theo. I.	$R,T,U,$	$P.$
$1\times R-1$	3	$R^{T+1}P-R^TP=R^TU-U$		
$3\div R^T-1$	4	$\frac{R^{T+1}P-R^TP}{R^T-1}=U$ or		
		$\frac{R^{T+1}-R^T}{R^T-1}\times P=\frac{R-1}{1-\frac{1}{R^T}}\times P=U$ Theo. (II.	$P,R,T,$	$U.$

From whence may the Theorems for T and R be deduced at Pleasure, as above.

Question 1. What is the present Worth P of 500 *l.* $=U$ yearly Rent, to continue $9=T$ Years abating at the Rate of 1.05 *l.* $=R$ the Amount of 1 *l.* *per Annum* Compound Interest?

Answer. $P=U\times\frac{1-\frac{1}{R^T}}{R-1}=3553.911$ *&c.* or 3553 *l.* 18 *s.* $2\frac{3}{4}$ *d.* near: Theo. I.

Question 2. What Yearly Rent $=U$ may be purchased for 3553 *l.* 18 *s.* $2\frac{3}{4}$ *d.* $=P$, to continue the Term of $9=T$ Years, allowing at the Rate of 1.05 *l.* $=R$ the Amount of 1 *l.* *per Annum* Compound Interest?

Answer. $U=P\times\frac{R-1}{1-\frac{1}{R^T}}=500$ *l. per* Theo. II.

Question 4. What is the present Worth $=P$ of 500 *l.* $=U$ yearly Rent, to continue $9=T$ Years, the first Year of which is not to commence till $5=t$ Years after the Payment is made, abating at the Rate of 1.05 *l.* $=R$ the Amount of 1 *l.* *per Annum* Compound Interest?

Here first find $P=U\times\frac{1-\frac{1}{R^T}}{R-1}$ or $\frac{R^TU-U}{R^{T+1}-R^T}=3553.911$, &c. *l.* the present Worth of the Annuity, supposing it to be immediately entered upon; but, according to the present Supposition, it is not to be entered upon till $5=t$ Years hence: Therefore the last Sum found, *viz.* 3553.911 *l.* is to be looked upon as the Amount of P for $5=t$ Years, at 1.05 *l.* *per Annum*, Compound Interest, *i. e.* $3553.911=A=PR^t$ (In. 596.) Whence $P=\frac{A}{R^t}$, or, reassuming the Value of A, $P=U\times\frac{1-\frac{1}{R^T}}{R^{t+1}-R^t}=2784.591$ *l.* or 2784 *l.* 11 *s.* $9\frac{3}{4}$ *d.* near the Price required. According to which last Theorem $U=P\times\frac{R^{t+1}-R^t}{1-\frac{1}{R^T}}$ from whence may be also raised Theorems for finding T and R in their order.

Corollary VI.

607. Also by summing up a Geometrical Series is learned to compute all the possible Permutations or Changes which can happen to any Number of Things n. For this has been shewn to be equal to the Series $n^1+n^2+n^3+n^4$, &c. till the last Term be n^n (In. 247.) But the Sum of that Geometrical Series whose first Term, common Multiplier, and Number of Terms is n, and greatest Term is n^n, is equal to $\frac{n^{n+1}-n}{n-1}$ (Theo. II. In. 603.)

Problem LIII.

608. In any Geometrical Series decreasing *ad infinitum*, so that the least Term l is taken for Nothing; from any two given of these three, *viz.* g, r, s, to find the third.

Effection.

Because $l=o$, therefore

				Given	Sought
	1	$sr-gr=s$	(In. 225.)		
$1\div\overline{s-g}$	2	$r=\frac{s}{s-g}$	Theorem I.	g, s,	r.
$1+gr-s$	3	$sr-s=gr$			
$3\div\overline{r-1}$	4	$s=\frac{gr}{r-1}$	Theo. II.	g, r,	s.
$3\div r$	5	$g=\frac{sr-s}{r}$	Theo. III.	r, s,	g.

SCHOLIUM VI.

609. By the foregoing Theorems may be answered the following Questions.

Question 1. Suppose a Body move 1 Mile the first Minute, half a Mile the second Minute, a Quarter of a Mile the third, $\frac{1}{8}$ of a Mile the fourth, $\frac{1}{16}$ of a Mile the fifth, *&c.* decreasing every following Minute to the one Half of what it moved the foregoing one, *ad infinitum.* It is required to determine, under this Suppofion, the Distance which the Body cannot exceed.

Here is given $g=1$, and $r=2$ to find s (Theo. II.) thus $s=\frac{gr}{r-1}=2$. Therefore the Body cannot exceed 2 Miles, tho' it be supposed to move to all Eternity.

Question 2. At what Rate r does that infinite Geometrical Series decrease, whose Sum is $s=40$, and the greatest Term is $32=g$?

Answer. $r=\frac{s}{s-g}=5$. (Theo. I.)

Question 3. What is the first or greatest Term g of that infinitely decreasing Geometrical Series whose Sum is $s=63$, and common Divisor is $7=r$?

Answer. $g=s\times\frac{r-1}{r}=54$. (Theo. III.)

COROLLARY VII.

610. Upon this last Problem is grounded the Method of computing the true Value P of *Freehold* or *Real Estates*, which are supposed to be purchased to continue for ever. For $P=U\times\frac{1-\frac{1}{R^T}}{R-1}$ (In. 606.) But in this Case T, and consequently R^T being infinite, therefore $\frac{1}{R^T}=0$. Whence $P=\frac{U}{R-1}$, $U=PR-P$, $R=\frac{U+P}{P}$.

Question 1. What is the present Worth of a Freehold Estate worth 500 *l.* $=U$ yearly Rent, abating at the Rate of 1.05 *l.* $=R$, or 5 *l. per Cent. per Annum, Compound Interest*?

Answer. $P=\frac{U}{R-1}=10000\,l.$

Question

Question 2. What is the present Worth P of a Freehold Estate worth 125 *l.* $=U$ Quarterly Rent, abating at the Rate of $\overline{1.05}^{\frac{1}{4}}$ or 1.01227223 *l.* $=R$ the Amount of 1 *l.* the Quarter, *Compound Interest.*

Answer $P=\frac{U}{R-1}=10183968$ *l.* or 10183 *l.* 19 *s.* $4\frac{1}{4}$ *d.* near.

CHAP. III.

Of Problems relating to Arithmetical and Geometrical Progression.

PROBLEM LIV.

611. IT is required to find three Terms in $\div$ from their Sum s, and Sum of their Squares z given.

Effection.

For the first Term put a, and for the common Difference e. Then is

	1	$a=$ the first Term.
	2	$a+e=$ the second Term.
	3	$a+2e=$ the third Term.
$1+2+3$	4	$3a+3e=s$ by the Question.
$4-3a$	5	$3e=s-3a$
$5\div3$	6	$e=\frac{s-3a}{3}$
$1+6$	7	$\frac{s}{3}=$ the second Term.
$7+6$	8	$\frac{2s-3a}{3}=$ the third Term.
$1\odot^{2}$	9	$aa=$ the Square of the first Term.
$7\odot^{2}$	10	$\frac{ss}{9}=$ the Square of the second Term.
$8\odot^{2}$	11	$\frac{4ss-12sa+9aa}{9}=$ the Square of the third Term.
$9+10+11$	12	$\frac{5ss-12sa+18aa}{9}=z$ by the Question.

82+9

12×9	13	$5ss-12sa+18aa=9z$
$13-5ss$	14	$-12sa+18aa=9z-5ss$
$14\div 18$	15	$-\frac{2}{3}sa+aa=\frac{9z-5ss}{18}$
$15+\frac{1}{9}ss$	16	$\frac{1}{9}ss-\frac{2}{3}sa+aa=\frac{9z-3ss}{18}=\frac{3z-ss}{6}$
$16\,w\,2$	17	$\frac{1}{3}s-a=\sqrt{\frac{3z-ss}{6}}$
$\therefore$	18	$\frac{1}{3}s-\sqrt{\frac{3z-ss}{6}}=a$ the first Term.
6, 18,	19	$\sqrt{\frac{3z-ss}{6}}=e$ the common difference.
$18+19$	20	$\frac{1}{3}s=$ the second Term.
$19+20$	21	$\frac{1}{3}s+\sqrt{\frac{3z-ss}{6}}=$ the last Term.

Ex. gr. If $s=18$ and $z=116$, then $a=\frac{1}{3}s-\sqrt{\frac{3z-ss}{6}}=4$ the first Term, $\frac{1}{3}s=6$ the second Term, and $\frac{1}{3}s+\sqrt{\frac{3z-ss}{6}}=8$ the last Term.

PROBLEM LV.

612. It is required to find three Terms in $\div$ whose Sum is s and their Product equal to the Sum of their Squares.

Effection.

Put a for the first Term, then will $\frac{s}{3}$ be the second, and $\frac{2s-3a}{3}$ is the third, according to which $\frac{2ssa-3saa}{9}$ is the Product of all the Terms, and $\frac{5ss-12sa+18aa}{9}$ is the Sum of their Squares, as in the last.

	1	$\frac{2ssa-3saa}{9}=\frac{5ss-12sa+18aa}{9}$
$\overline{1}\times 9$	2	$2ssa-3saa=5ss-12sa+18aa$
Whence	3	$\frac{ss+6s-\overline{ssss-3sss-54ss}\vert^{\frac{1}{2}}}{3s+18}=a$ the first Term.
	4	$\frac{s}{3}=$ the second Term.
4−3	5	$\frac{\overline{ssss-3sss-54ss}\vert^{\frac{1}{2}}}{3s+18}=$ the common Difference.
4+5	6	$\frac{ss+6s+\overline{s^4-3s^3-54ss}\vert^{\frac{1}{2}}}{3s+18}=$ the last Term.

From the last Step it appears that for s can be assumed no Number whose Square is less than 54, because thus $\overline{s^4-3s^3-54ss}\vert^{\frac{1}{2}}$ will be impossible, as supposing the Square Root of a defective Quantity.

If $s=10$, then is the Term $\frac{15}{6}$, the second $\frac{20}{6}$, and the third Term $\frac{25}{6}$ whose Product and Sum of their Square is $\frac{1250}{36}$.

PROBLEM LVI.

613. To find four Terms in $\div$, whose Sum is s, and Sum of their Squares z.

Effection.

Put a for the first Term, and $e=$ the common Difference. Then.

	1	$a=$ the first Term.
	2	$a+e=$ the second Term.
	3	$a+2e=$ the third Term.
	4	$a+3e=$ the fourth Term.
1+2+3+4	5	$4a+6e=s$ by the Question.
Whence	6	$e=\frac{s-4a}{6}$
1+6	7	$\frac{s+2a}{6}=$ the second Term.
7+6	8	$\frac{2s-2a}{6}=$ the third Term.
8+6	9	$\frac{3s-6a}{6}=$ the fourth Term.

$1 \odot^2$	10	$aa = \frac{36aa}{36} =$
$7 \odot^2$	11	$\frac{ss + 4sa + 4aa}{36}$
$8 \odot^2$	12	$\frac{4ss - 8sa + 4aa}{36}$
$9 \odot^2$	13	$\frac{9ss - 36sa + 36aa}{36}$
$10+11+12+13$	14	$\frac{14ss - 40sa + 80aa}{36} = z$ by the Queſtion.
Whence	15	$a = \frac{1}{4}s - \overline{\frac{36z - 9ss}{80}}\Big\|^{\frac{1}{2}}$ the firſt Term.
6.	16	$e = \frac{2}{3}\overline{\frac{36z - 9ss}{80}}\Big\|^{\frac{1}{2}} = \overline{\frac{36z - 9ss}{180}}\Big\|^{\frac{1}{2}}$ the common Difference.

Ex. gr. If $s = 28$, and $z = 216$, then $a = \frac{1}{4} - \overline{\frac{36z - 9ss}{80}}\Big|^{\frac{1}{2}} = 4$, $e = \overline{\frac{36z - 9ss}{180}}\Big|^{\frac{1}{2}} = 2$. Conſequently the other three Terms are 6, 8, 10.

Problem LVII.

614. To find five Terms in ∻, *viz.* a, $a+e$, $a+2e$, $a+3e$, $a+4e$, whereof the firſt Term is in Proportion to the laſt as b to d, and the Sum of all the Terms is equal to the Square of the Middle or third Term.

Effection.

	1	$a : a + 4e = b : d$ } *By the Queſtion.*
Or	2	$ad = ab + 4be$
Whence	3	$a \times \frac{d - b}{4b} = e$
	4	$a = \frac{4ab}{4b}$ the firſt Term by the Queſtion.
$4+3$	5	$\frac{3ab + ad}{4b} =$ the ſecond Term.
$5+3$	6	$\frac{2ab + 2ad}{4b} =$ the third Time.

6+3	7	$\frac{ab+3ad}{4b}=$ the fourth Term.
7+3	8	$\frac{4ad}{4b}=$ the fifth Term.
4+5+6+7+8	9	$\frac{5ab+5ad}{2b}=$ the Sum of all the Terms.
6 ⊙ 2	10	$\frac{4a^2b^2+8a^2bd+4a^2d^2}{16bb}=$ the Square of the third Term.
	11	$\frac{5ab+5ad}{2b}=\frac{4a^2b^2+8a^2bd+4a^2d^2}{16bb}$ by the Question.
Whence	12	$a=\frac{10\times\overline{bb+dd}}{b^2+2bd+d^2}$
8, 12,	13	$e=a\times\frac{d-b}{4b}=\frac{5\times\overline{dd-bb}}{2bb+4bd+2dd}$ or $\frac{10\times\overline{dd+bb}}{4\times\overline{b+d}^2}$

Ex. gr. If $b=3$, and $d=5$, then will the five Terms required be $\frac{90}{24}$, $\frac{105}{24}$, $\frac{120}{24}$, $\frac{135}{24}$, $\frac{150}{24}$ or $\frac{30}{8}$, $\frac{35}{8}$, $\frac{40}{8}$, $\frac{45}{8}$, $\frac{50}{8}$.

Problem LVIII.

615. To find three Numbers in ∺ a, e, y, whose Sum is s, and Sum of their Squares z.

Effection.

	1	$ay=ee$ (In. 191.)
1×2	2	$2ay=2ee$
	3	$a+e+y=s$ by the Question.
3−e	4	$a+y=s-e$
4 ⊙ 2	5	$a^2+2ay+yy=ss-2se+ee$
5−2	6	$a^2+y^2=ss-2se-ee$
6+ee	7	$a^2+e^2+y^2=ss-2se+ee=z$ by the Question.
Whence	8	$e=\frac{ss-z}{2s}$ Theo. I.
4, 8,	9	$a+y=s-e=\frac{ss+z}{2s}$
9−y	10	$a=\frac{ss+z}{2s}-y$

1÷7

$1 \div y$	11	$a = \frac{ee}{y} = \frac{s^4 - 2ssz + zz}{4ssy}$
10, 11	12	$\frac{ss+z}{2s} - y = \frac{s^4 - 2s^2z + z^2}{4ssy}$
Whence	13	$y = \frac{\overline{10ssz - 3ssss - 3zz}^{\frac{1}{2}} + ss + z}{4s}$ Theo. II.
11	14	$a = \frac{ssss - 2ssz - zz}{s \times \overline{10ssz - 3ssss - 3zz}^{\frac{1}{2}} + ss + z}$ Theo. III.

Ex. gr. If $s=228$, $z=19152$, $e=72$, $y=108$, $a=48$.

Problem LIX.

616. To find three Quantities in Geometrical Progreſſion b, e, y from the firſt given equal b, and the Sum of the Squares of the other $ee+yy=z$.

Effection.

	1	$ee+yy=z$ by the Queſtion.
$1-yy$	2	$ee=z-yy$
	3	$ee=by$ (In. 191.)
2, 3,	4	$z-yy=by$
Whence	5	$y=\frac{\overline{4z+bb}^{\frac{1}{2}}-b}{2}$ Theo. I.
3,	6	$e=\overline{\frac{\overline{4bbz+bbbb}^{\frac{1}{2}}-bb}{2}}^{\frac{1}{2}}$ Theo. II.

Ex. gr. If $b=48$, and $z=16848$, then $y=108$, and $e=72$.

Problem LX.

617. To find three Quantities in ∺ a, e, y, whoſe Sum is s, and the Sum of the Squares of the Extreams z.

Effection.

	1	$ay=ee$ (In. 191.)
1×2	2	$2ay=2ee$
	3	$a+e+y=s$ by the Queſtion.
$3-e$	4	$a+y=s-e$

$4\odot^2$	5	$a^2+2ay+yy=ss-2se+ee$.
$5-2$	6	$a^2+y^2=ss-2se-ee=z$ by the Queſtion.
Whence	7	$e=\overline{2ss-z}^{\frac{1}{2}}-s$
	8	$y=\frac{\overline{ss-2se-ee}^{\frac{1}{2}}+s-e}{2}$
	9	$a=s-e-y$.

Ex. gr. If $s=228$, $z=13968$, then $e=72$, $y=108$, $a=48$.

Problem XLI.

618. To find three Quantities a, e, y in $\div\!\div$, whereof the Difference of the Extreams $y-a=d$, and the Sum of all their Squares $a^2+e^2+y^2=z$.

Effection.

Make	1	$y+a=A$
	2	$y-a=d$ by the Queſtion.
Then	3	$y=\frac{A+d}{2}$ In. 546.
	4	$a=\frac{A-d}{2}$
3×4	5	$ay=\frac{AA-dd}{4}=ee$ (In. 191.)
$3\odot^2$	6	$\frac{AA+2Ad+dd}{4}=yy$
$4\odot^2$	7	$\frac{AA-2Ad+dd}{4}=aa$
$5+6+7$	8	$\frac{3AA+dd}{4}=a^2+e^2+y^2=z$ by the Queſtion.
Whence	9	$A=y+a=\overline{\frac{4z-dd}{3}}^{\frac{1}{2}}$
3,	10	$y=\frac{\overline{\frac{4z-dd}{3}}^{\frac{1}{2}}+d}{2}$ Theo. I.

4, [11]

4,	11	$a=\dfrac{\sqrt{\dfrac{4z-dd}{3}}-d}{2}$	Theo. II.
$5w^2$	12	$e=\sqrt{\dfrac{4z-4dd}{12}}$	Theo. III.

Ex. gr. If $d=60$, and $z=19152$, then $y=108$, $a=48$, $e=72$.

Problem LXII.

619. To find three Quantities in $\div\!\div$, a, e, y, whereof the Difference of the Extreams $y-a=d$, and the Difference of the Squares of the Extreams $yy-aa$ is in Proportion to the Sum of all the Squares $a^2+e^2+y^2$, as p to q.

Effection.

Make	1	$y+a=A$	
	2	$y-a=d$	by the Question.
Then	3	$y=\dfrac{A+d}{2}$	In. 546.
	4	$a=\dfrac{A-d}{2}$	
1×2	5	$yy-aa=Ad$	
$3 \circledS^2$	6	$yy=\dfrac{AA+2Ad+dd}{4}$	
$4 \circledS^2$	7	$aa=\dfrac{AA-2Ad+dd}{4}$	
3×4	8	$ee=ay=\dfrac{AA-dd}{4}$	(In. 191.)
6+7+8	9	$\dfrac{3AA+dd}{4}=a^2+e^2+y^2$	
		$yy-aa:a^2+e^2+y^2=p:q$	by the Question.
	10	$Ad:\dfrac{3AA+dd}{4}=p:q$	
Whence	11	$A=\dfrac{\sqrt{4ddqq-3ppdd}+2dq}{3p}$	Theorem.

3, |12|

3,	12	$y=\frac{A+d}{2}$
4,	13	$a=\frac{A-d}{2}$
$8ww^{2}$	14	$e=\overline{ay}^{\frac{1}{2}}=\overline{\frac{A^2-d^2}{4}}^{\frac{1}{2}}$

Ex. gr. If $d=30$, $p=5$, and $d=7$, then $A=50$, $y=40$, $a=10$, $e=20$.

Problem LXIII.

620. To find four Quantities in $\div\!\!\div$ a, e, y, u, whereof the Sum of the Extreams $a+u=S$, and the Sum of the Means $e+y=s$.

Effection.

	1	$a+u=S$ by the Queſtion.
$1-a$	2	$u=S-a$ the greater Extream.
	3	$e+y=s$ by the Queſtion.
$3-e$	4	$y=s-e$ the greater Mean.
	5	$\left\{\begin{array}{l} a:e=e:\overline{s-e} \\ a\times\overline{s-e}=ee \end{array}\right\}$ (In. 191.)
$5\div\overline{s-e}$	6	$a=\frac{ee}{s-e}$ the leſſer Extream.
	7	$\left\{\begin{array}{l} e:\overline{s-e}=\overline{s-e}:u \\ eu=ss-2se+ee \end{array}\right\}$ (In. 191.
$7\div e$	8	$u=\frac{ss-2se+ee}{e}$
$6+8$	9	$a+u=\frac{ee}{s-e}+\frac{ss-2se+ee}{e}=S$ by the Queſtion.
Whence	10	$e=\frac{s-\overline{\frac{Sss-sss}{S+3s}}^{\frac{1}{2}}}{2}$ Theorem.
6,	11	$a=\frac{ee}{s-e}$
4,	12	$y=s-e$
2,	13	$u=S-a$

Ex. gr. If $S=175$, $s=150$, then $e=60$, $a=40$, $y=90$, and $u=135$.

Pro-

Problem. LXIV.

621. To find four Quantities in ∺ a, e, y, u, whereof the Difference of the Extreams $u-a=D$, and the Difference of the Means $y-e=d$.

Effection.

	1	$u-a=D$ by the Question.
$1+a$	2	$u=D+a$
	3	$y-e=d$ by the Question.
$3+e$	4	$y=d+e$
	5	$\left\{\begin{matrix} a:e=e:\overline{d+e} \\ ee=a\times\overline{d+e} \end{matrix}\right\}$ (In. 191.)
$5\div\overline{d+e}$	6	$a=\frac{ee}{d+e}$
	7	$\left\{\begin{matrix} e:\overline{d+e}=\overline{d+e}:u \\ ue=dd+2de+ee \end{matrix}\right\}$ (In. 191.)
$7\div e$	8	$u=\frac{dd+2de+ee}{e}$
$8-6$	9	$u-a=\frac{dd+2de+ee}{e}-\frac{ee}{d+e}=D$ by the Question.
Whence	10	$e=\frac{\overline{\frac{Ddd+ddd}{D-3d}}^{\frac{1}{2}}-d}{2}$ Theorem.
6,	11	$a=\frac{ee}{d+e}$
4,	12	$y=e+d$
2,	13	$u=a+D$

Ex. gr. If $D=95$, $d=30$, then $e=60$, $a=40$, $y=90$, and $u=135$.

Scolium VII.

622. In the Solution of Questions in ∺ the Work will oftentimes be much ſhortned by Help of the following Theorems, which may therefore be raiſed for that Purpoſe from In. 190, 191, 223, 224. Where obſerve that a is always put for the leaſt Term, and the other Letters for the other Terms in their Order.

I.

Let a, e, y represent three Quantities in ∺.

Then	1	$ee=ay$ (In. 191.)	
$1\times e$	2	$eee=aey$	Theo. I.
	3	$aa+yy=aa+yy$	
1×2	4	$2ee=2ay$	
$3+4$	5	$a^2+2ee+yy=\overline{a+y}^2=aa+2ay+yy$	Theo. II.
	6	$aa=aa$	
$1-6$	7	$ee-aa=a\times\overline{y-a}$	Theo. III.
	8	$yy=yy$	
$8-1$	9	$yy-ee=y\times\overline{y-a}$	Theo. IV.
$1+6$	10	$ee+aa=a\times\overline{y+a}$	Theo. V.
$8+1$	11	$yy+ee=y\times\overline{y+a}$	Theo. VI.
$11\times e$	12	$yye+eee=ye\times\overline{y+a}$	
∴	13	$yy+ee:ye=y+a:e$ (In. 190.)	Theo. VII.
$10\times e$	14	$eee+aae=ae\times\overline{y+a}$	
∴	15	$ee+aa:ae=y+a:e$ (In. 190.)	Theo. VIII.
∴ 13, 15	16	$yy+ee:ye=ee+aa:ae$	
Consequently	17	$yy+ee:ee+aa=y:a$	Theo. IX.
$9\times e$	18	$yye-eee=ye\times\overline{y-a}$	
∴	19	$yy-ee:ye=y-a:e$	Theo. X.
$7\times e$	20	$eee-aae=ae\times\overline{y-a}$	
∴	21	$ee-aa:ae=y-a:e$	Theo. XI.
∴ 19, 21	22	$yy-ee:ye=ee-aa:ae$	
Or	23	$yy-ee:ee-aa=y:a$	Theo. XII.
17, 23,	24	$yy+ee:yy-ee=ee+aa:ee-aa$	Theo. XIII.
	25	$yy=yy$	
4	26	$0=-2ay+2ee$	
	27	$0=aa-aa$	
	28	$-aa=-aa$	
$25+26+27+28$	29	$yy-aa=\overline{y-a}^2+2\times\overline{ee-aa}$	Theo. XIV.
	30	$-ee=-ee$	
$25+26+27+30$	31	$yy-ee=\overline{y-a}^2+ee+aa$	Theo. XV.

$1\times a$	32	$aee=yaa$	
Whence	33	$ee:aa=y:a$	Theo. XVI.
17, 33,	34	$yy+ee:ee+aa=ee:aa=y^2:ee$	Theo. XVII.
23, 33	35	$yy-ee:eee-aa=ee:aa=y^2:e^2$	Theo. XVIII.
$1\times y$	36	$yee=yya$	
Whence	37	$y^2:e^2=y:a$	Theo. XIX.
	38	$+ye=+ye$	
38—1	39	$e\times\overline{y-e}=y\times\overline{e-a}$	
Whence	40	$y-e:e-a=y:e=e:a$	Theo. XX.
	41	$y^3+a^3=y^3+a^3$	
$1\times 3a$	42	$3aee=3aay$	
$1\times 3y$	43	$3yee=3ayy$	
41+42+43	44	$y^3+a^3+3ee\times\overline{y+a}=\overline{y+a}^3$	Theo. XXI.

II.

Let a, e, y, u reprefent four Quantities in $\div\div$.

	1	$au=ey$ (In. 190.)	
	2	$ay=ee$ (In. 191.)	
	3	$eu=yy$ (In. 191.)	
	4	$ey=ey$	
1+2+3+4	5	$\overline{e+a}\times\overline{u+y}=yy+2ye+ee=\overline{y+e}^2$	
Whence	6	$u+y:y+e=y+e:e+a$	Theo. XXII.
	7	$ae=ae$	
2,	8	$ee=ay$	
	9	$ye=ye$	
4	10	$ue=yy$	
7+8+9+10	11	$\overline{a+e+y+u}\times e=\overline{a+y}\times\overline{e+y}$	
Whence	12	$a+e+y+u:e+y=a+y:e$	Theo. XXIII.
Theo. VII.	13	$a+y:e=ee+yy:ey$ or au	Theo. XXIV.
∴	14	$a+e+y+u:e+y=ee+yy:ey$ or au	Theo. XXV.
$2\odot^2$	15	$aayy=e^4$	
$1\odot^2$	16	$aauu=eeyy$	
	17	$eeyy=eeyy$	
$4\odot^2$	18	$eeuu=y^4$	
15+16+17+18	19	$\overline{aa+ee}\times\overline{yy+uu}=\overline{ee+yy}^2$	
∴	20	$aa+ee:ee+yy=ee+yy:yy+uu$	Theo. XXVI.
	21	$ee=ee$	

$\overline{1\times2}$	22	$2ey=2au$	
	23	$yy=yy$	
$\overline{1\times2}$	24	$o=2au-2ey$	
	25	$\overline{e+y}^2=\overline{e-y}^2+4au$ or $+4ey$ (Step 1.)	Theo. XXVII.
	26	$aae=aae$	
$2\times e$	27	$eee=yae$	
$1\times y$	28	$yye=ayu$	
$4\times u$	29	$uue=yyu$	
$26+27+28+29$	30	$\overline{aa+ee+yy+uu}\times e=\overline{ae+yu}\times\overline{a+y}$	
$\therefore$	31	$\overline{a^2+e^2+y^2+u^2}:\overline{ae+yu}=\overline{a+y}:e$	Theo. XXVIII.
But	32	$\overline{a+e+y+u}:\overline{e+y}=\overline{a+y}:e$	(Step. 12.)
$\therefore$	33	$a^2+e^2+y^2+u^2:a+e+y+u=ae+yu:e+y$	Th. XXIX.
Step. 14	34	$a^2+e^2+y^2+u^2:ee+yy=ae+yu:au$ or ey	Theo. XXX.
	35	$\frac{\overline{a+e+y+u}^2-a^2-e^2-y^2-u^2}{2}=ae+ay+au+ey+eu+yu$	
Subst.	36	ee for ay, and yy for ey Step. 2d. and 4th.	
Then	37	$\frac{\overline{a+e+y+u}^2-a^2-e^2-y^2-u^2}{2}=ae+au+yu+ee+ey+yy$	
	38	$ee+2ey+yy=\quad ee+2ey+yy$	
$37-38$	39	$\frac{\overline{a+e+y+u}^2-a^2-e^2-y^2-u^2}{2}-\overline{e+y}^2=ae+au+yu-ey$	
1	40	$o=-au+ey$	
$39+40$	41	$\frac{\overline{a+e+y+u}^2-a^2-e^2-y^2-u^2}{2}-\overline{e+y}^2=ae+yu$	Theo. (XXXI.
33.	42	$ae+yu=\frac{\overline{aa+ee+yy+uu}\times\overline{e+y}}{a+e+u+y}$	
	43	$\frac{\overline{a+e+y+u}^2-a^2-e^2-y^2-u^2}{2}-\overline{e+y}^2:a^2+e^2+y^2+u^2=e+y:a+e+u+y$	Theo. XXXII.
	44	$aey=eee=aau$ } Step. I.	Theo. XXXIII.
	45	$eyu=yyy=auu$ } Step. I.	Theo. XXXIV.
$44+45$	46	$\overline{a+u}\times eu=e^3+y^3=\overline{a+u}\times au$	Theo. XXXV.

because

		because $au=ey$ Step. 1.	
	47	$\overline{a+u}^3=a^3+3a^2u+3auu+u^3$	
46×3	48	$3eee+3yyy=3aau+3auu$	
Then	49	$\overline{a+u}^3=a^3+u^3+3\times\overline{e^3+y^3}$	Theo. XXXVI.
45—44	50	$y^3-e^3=\overline{u-a}\times au=auu-aau.$	
	51	$\overline{u-a}^3=u^3-3uua+3uaa-a^3$	
50×3	52	$3y^3-3e^3=3uua-3uaa$	
51+52	53	$\overline{u-a}^3+3\overline{y^3-e^3}=u^3-a^3$	Theo. XXXVII.
	54	$3a^2y+a^2u=a^2\times\overline{3y+a}$	
Subst.	55	*ee* for *ay*, and *ey* for *au* (Step. 1, 2.)	
Then	56	$3aee+aey=a^2\times\overline{3y+u}$	
Subst.	57	*eee* for *aey* (Step. 27.)	
Then	58	$3aee+eee=a^2\times\overline{3y+u}$	
Whence	59	$ee:aa=3y+u:3a+e$	
But	60	$e^2:a^2=y^2:e^2=u^2:y^2$	
∴	61	$e^2:a^2=y^2:e^2=u^2:y^2=3y+u:3a+e$	Theo. XXXVIII.
	62	$a^3+3a^2e=a^3+3a^2e=a^2\times\overline{a+3e}$	
58+62	63	$a^3+3a^2e+3ae^2+e^3=a^2\times\overline{3y+u+a+3e}$	
Or		$\overline{a+e}^3=a^2\times\overline{u+a+3\times\overline{y+e}}$	Theo. XXXIX.
44	64	$eee=aau$	
1×3e	65	$3eey=3aeu$	
1×3y	66	$3eyy=3ayu$	
45	67	$yyy=auu$	
64+65+66+67	68	$\overline{e+y}^3=au$ or $ey\times\overline{a+u+3\times\overline{e+y}}$	Theo. XL.
	69	$a^3=aaa$	
44	70	$e^3=aau$	
45	71	$y^3=auu$	
	72	$u^3=uuu$	
69+70+71+72	73	$a^3+e^3+y^3+u^3=\overline{a+u}\times\overline{a^2+u^2}$	Theo. XLI.
44×a	74	$aaau=aeee$	
45×u	75	$auuu=uyyy$	
4×ae	76	$0=aeyy-aeeu$	
2×uy	77	$0=uyee=uyya$	

74+75+76+77	78	$aux\overline{aa+uu}=ae^3+aey^2+uye^2+uy^3-aue^2-auy^2$	
Whence	79	$a^2+u^2:e^2+y^2=ae+uy-au$ or $-ey:au$ or ey	Theo. (XLII.

III.

Let, a, e, y, u, b, reprefent five Quantities in ∺.

	1	$ay=ee$ } (In. 191.)	
	2	$by=uu$	
1+2	3	$y\times\overline{a+b}=ee+uu$	Theo. XLIII.
2—1	4	$y\times\overline{b-a}=uu-ee$	
Whence	5	$b-a:u-a=u+a:y$	Theo. XLIV.
	6	$ab=eu$ (In. 223.)	
1+6	7	$a\times\overline{b+y}=e\times\overline{u+e}$	
Whence	8	$a:e=u+e:b+y=e:y=y:u$	Theo. XLV.
	9	$ya=ee$	
	10	$yy=yy$	
	11	$yb=uu$	
9+10+11	12	$y\times\overline{b+y+a}=uu+yy+ee$	Theo. XLVI.
	13	$ay=ee$	
	14	$by=uu$	
	15	$be=yu$	
	16	$ye=au$	
13+14+15+16	17	$e\times\overline{y+b}+y\times\overline{a+b}=e^2+u\times\overline{a+y+u}$	Theo. XLVII.
11×y	18	$byy=yuu$	
	19	$yyy=auu$ (Theo. XXXIV.)	
18+19	20	$yy\times\overline{b+y}=uu\times\overline{y+a}$	
Whence	21	$yy:uu=y+a:b+y$	Theo. XLVIII.
2 ☉ ^{2}	22	$yybb=uuuu$	
1 ☉ ^{2}	23	$yyaa=eeee$	
22—23	24	$yy\times\overline{bb-aa}=\overline{uu+ee}\times\overline{uu-ee}$	
Whence	25	$bb-aa:uu-ee=uu+ee:yy$	Theo. XLIX.
3×y	26	$yy\times\overline{a+b}=y\times\overline{ee+uu}$	
Whence	27	$a+b:y=uu+ee:yy$	
25, 27.	28	$bb-aa:uu-ee=a+b:y$	Theo. L.

$ya=ee$

	29	$ya=ee$	
	30	$2yy=2eu$	
	31	$yb=uu$	
29+30+31	32	$y\times\overline{a+2y+b}=\overline{u+e}^2$	
Whence	33	$y:u+e=u+e:a+2y+b$	Theo. LI.
	34	$b^2+a^2=b^2+a^2$	
3,	35	$uu+ee=yb+ya$	
	36	$yy=ab$ (In. 224.)	
34+35+36	37	$b^2+u^2+y^2+e^2+a^2=b\times\overline{b+y+a}+a\times\overline{y+a}$	Theo. LII.
	38	$b^3=b^3$	(In. 190, 223, 224.)
	39	$u^3=byu=bbe$	
	40	$y^3=euy=bee$	
	41	$e^3=aye=aau$	
	42	$a^3=b^3$	
38+39 &c.	43	$b^3+u^3+y^3+e^3+a^3=b\times\overline{bb+be+ee}+aa\times\overline{u+a}$	Theo. (LIII.
34,	44	$b^4+a^4=b^4+a^4$	
35	45	$u^4+e^4=y^2b^2+y^2a^2$	
36	46	$y^4=a^2b^2$	
37.	47	$b^4+u^4+y^4+e^4+a^4=b^2\times\overline{b^2+y^2+a^2}+a^2\times\overline{y^2+a^2}$	(Theo. LIV.

After the same Manner may innumerable other Theorems be raised, which the Learner may pursue at his own Discretion, whose Use will be seen in the Effection of the following Problems.

PROBLEM LXV.

623. To find three Quantities in $\div\!\div$, a, e, y, whereof the Sum of the Extreams $a+y=s$, and the Sum of the Cubes of the said Extreams $a^3+y^3=m$.

Effection.

	1	$m+3ees=sss$	(Theo. XXI.)
Whence	2	$e=\left\lvert\dfrac{sss-m}{3s}\right.^{\frac{1}{2}}$. the Mean.	
2 $\odot^2$	3	$ay=ee=\dfrac{sss-m}{3s}$	(In. 191.)

	4	$a+y=s$ by the Queſtion.
Subſt.	5	$\frac{sss-m}{3s}$ for p (In 547.)
Then	6	$y=\frac{s+\overline{\frac{4m-sss}{3s}}^{\frac{1}{2}}}{2}$ (by Step. 8. there) the greater Extream.
And	7	$a=\frac{s-\overline{\frac{4m-sss}{3s}}^{\frac{1}{2}}}{2}$ (by Step. 10. there) the leſſer Extream.

Ex. gr. If $s=5$, $m=65$, then $e=2$, $y=4$, $a=1$.

Problem LXVI.

624. To find three Quantities in $\div\div$, whoſe Sum is z, and Sum of their Cubes m.

Effection.

	1	$a+e+y=z$ } by the Queſtion.
	2	$a^3+e^3+y^3=m$ } by the Queſtion.
$1-e$	3	$a+y=z-e$
$2-eee$	4	$a^3+y^3=m-e^3$
	5	$m-e^3+3e^2\times\overline{z-e}=\overline{z-e}^3$ (Theo. XXI.)
Whence	6	$zze-eee=\frac{zzz-m}{3}$
Subſtit.	7	$ay=ee=p$ (In. 191.)
Subſtit.	8	$a+y=z-e=s$
Whence	9	$y=\frac{s+\overline{ss-4p}^{\frac{1}{2}}}{2}$ (In. 547.)
	10	$a=\frac{s-\overline{ss-4p}^{\frac{1}{2}}}{2}$ (In. 547.)

Ex. gr. If $z=14$, and $m=584$, then $e=4$ (Step. 6.) $y=8$, $e=2$.

Problem LXVII.

625. To find four Quantities in $\div\div$, a, e, y, u, whoſe Sum is s, and the Sum of their Squares z.

Effection.

Effection.

For the Sum of the Means $e+y$ put A, then will the Sum of the Extreams $a+u$ be $s-A$.

Then	1	$\frac{ss-z}{2}-AA : z = A : s$ (Theo. XXXII.)
Whence	2	$A=\frac{\overline{2ssss-2ssz+zz}^{\frac{1}{2}}-z}{2s}$
Subftit.	3	$a+u=s-A=S$
Confequently	4	$e=\frac{A-\frac{SAA-AAA}{s+3A}}{2}$ (In. 620.)
	5	$a=\frac{ee}{A-e}$
	6	$y=A-e$
$3-a$	7	$u=S-a$.

Ex. gr. If $s=325$, and $z=31525$, then $A=150$, $S=175$, $e=60$, $a=40$, $y=90$, $u=135$.

PROBLEM LXVIII.

626. To find four Quantities in $\div\!\div$, a, e, y, u, the Sum of whose Cubes is m, and the Sum of the Extreams $a+u=s$.

Effection.

Put	1	$z=a^2+u^2$
Then	2	$m=sz$ (Theo. XLI.)
$2\div s$	3	$z=\frac{m}{s}$
$1-u^2$	4	$a^2=z-u^2$
	5	$a+u=s$ by the Queftion.
$5-u$	6	$a=s-u$
6 ⓐ ^{2}	7	$a^2=ss-2su+uu$
4, 7,	8	$ss-2su+uu=z-uu$
Whence	9	$u=\frac{\overline{2z-ss}^{\frac{1}{2}}+s}{2}$
6,	10	$a=s-u$
	11	$e=\overline{aau}^{\frac{1}{3}}$ Theo. XXXIII.
	12	$y=\overline{auu}^{\frac{1}{3}}$ Theo. XXXIV.

Ex.

Ex. gr. If $m=585$, and $s=9$, then $z=65$, and $u=8$, $a=1$, $e=2$, $y=4$.

Problem LXIX.

627. To find five Quantities in ∺, a, e, y, u, b, whereof $a+b=b$, and $e+y+u=c$.

Effection.

	1	$a+b=b$ } by the Question.
	2	$e+y+u=c$ }
2—y	3	$e+u=c-y$
	4	$y:c-y=c-y:b+2y$ (Theo. LI.)
Whence	5	$y=\frac{\overline{bb+4bc+8cc}^{\frac{1}{2}}-b-2c}{2}$
Substit.	6	$e+u=c-y=d$
6—u	7	$e=d-u$
	8	$eu=yy$ (In. 191.)
8÷u	9	$e=\frac{yy}{u}$
7, 9,	10	$d-u=\frac{yy}{u}$
Whence	11	$u=\frac{\overline{dd-4yy}^{\frac{1}{2}}+d}{2}$
7,	12	$e=d-u$
	13	$ay=ee$ (In. 191.)
13÷y	14	$a=\frac{ee}{y}$
	15	$ab=eu$ (In. 223.)
15÷a	16	$b=\frac{eu}{a}$

Ex. gr. If $b=17$, and $c=14$, then $y=4$, $d=10$, $u=8$, $e=2$, $a=1$, $b=16$.

Problem LXX.

628. To find five Quantities in ∺, a, e, y, u, b, whereof $a+b=b$, and $ee+yy+uu=z$.

Effection.

Effection.

	1	$a+b=b$ } by the Question.
	2	$e^2+y^2+u^2=z$ } by the Question.
$2-yy$	3	$ee+uu=z-yy$
	4	$by=z-yy$ (Theo. XLIII.
Whence	5	$y=\frac{\overline{bb+4z}^{\frac{1}{2}}-b}{2}$
	6	$ab=yy$ (In. 191.)
$6\div b$	7	$a=\frac{yy}{b}$
$1-b$	8	$a=b-b$
7, 8,	9	$\frac{yy}{b}=b-b$
Whence	10	$b=\frac{\overline{bb-4yy}^{\frac{1}{2}}+b}{2}$
8,	11	$a=b-b$
	12	$ee=ay$ (In. 191.)
$12\,w^2$	13	$e=\overline{ay}^{\frac{1}{2}}$
	14	$uu=by$ (In. 191.)
$14\,w^2$	15	$u=\overline{by}^{\frac{1}{2}}$

Ex. gr. If $b=17$, $z=84$, $y=4$, $b=16$, $a=1$, $e=2$, and $u=8$.

CHAP IV.

Of the ſumming up Ranks of homologous Polygons or Powers whoſe Roots or Sides are in Arithmetical Progreſſion, beginning with Unity.

Problem LXXI.

629. TO raiſe a *Polygon* P from its Root or Side n given: Or, which is the ſame, to find the Sum s of any Rank of Numbers in $\div$, whoſe leaſt Term l equals Unity, common Difference is d, and Number of Terms is n.

Effection.

Effection.

Here because $l=1$, therefore s or $P=\frac{2n+d\times\overline{nn-n}}{2}$ (In. 599. Theo. IX.) Hence, if the Number be a.

$$\left.\begin{array}{l} \textit{Lateral} \\ \textit{Trigon} \\ \textit{Tetragon} \\ \textit{Pentagon} \\ \textit{Hexagon} \\ \textit{Heptagon} \\ \textit{Octogon} \\ \textit{\&c.} \end{array}\right\} \text{i. e. if } d=\left\{\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \textit{\&c.} \end{array}\right\} \text{then } P=\left\{\begin{array}{l} n \\ \frac{nn+n}{2} \\ nn \\ \frac{3nn-n}{2} \\ 2nn-n \\ \frac{5nn-3n}{2} \\ 3nn-2n \\ \textit{\&c.} \end{array}\right.$$

COROLLARY VIII.

630. Therefore the Scale of *Polygons*, whose common Root is n, is a Series of Terms in $\div$, whose first Term is n, common Difference is $\frac{nn-n}{2}$ and Number of Terms is $d+1$, and consequently the Sum s of such a Series $=\frac{4n+dd\times\overline{nn-n}+d\times\overline{nn+3n}}{4}$ (In. 599. Theorem IX.) Whence, if the highest Term be a

$$\left.\begin{array}{l} \textit{Lateral} \\ \textit{Trigon} \\ \textit{Tetragon} \\ \textit{Pentagon} \\ \textit{Hexagon} \\ \textit{Heptagon} \\ \textit{Octogon} \\ \textit{\&c.} \end{array}\right\} \text{i. e. if } d=\left\{\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \textit{\&c.} \end{array}\right\} \text{then } s=\left\{\begin{array}{l} n \\ \frac{nn+3n}{2} \\ \frac{3nn+3n}{2} \\ 3nn+n \\ 5nn \\ \frac{15nn-3n}{2} \\ \frac{21nn-7n}{2} \\ \textit{\&c.} \end{array}\right.$$

LEMMA

Lemma *to the following Problem.*

631. If a Series of *Laterals* beginning with a Cypher (0, 1, 2, 3, 4, 5, *&c.*) be placed in the contrary order, (*&c.* 5, 4, 3, 2, 1, 0,) then beginning with the first Terms of each to the left Hand, I say the greatest *Trigon*, arising from summing up the former Series, will be equal to the greatest *Trigon* raised by summing up the latter: The greatest *first Pyramidal* raised by summing up the former Series will be Subduple or $\frac{1}{2}$ of that raised from the latter: The greatest *second Pyramidal* raised from the former Subtriple or $\frac{1}{3}$ of that raised from the latter: The greatest *third Pyramidal* raised from the former Subquadruple or $\frac{1}{4}$ of that raised from the latter: The greatest *fourth Pyramidal* raised from the former Subquintuple or $\frac{1}{5}$ of that raised from the latter, *&c. ad Infinitum.*

Demonstration.

Put	*n*	to represent the Series of	*Laterals*	or the Sums of so many	*Units*	as there are Terms in the Arithmetical Series.
	p		*Trigons*		*Laterals*	
	q		*1st Pyramidals*		*Trigons*	
	r		*2d Pyramidals*		*1st Pyram.*	
	s		*3d Pyramidals*		*2d Pyram.*	
	t		*4th Pyramidals*		*3d Pyram.*	
	&c.		*&c.*		*&c.*	

Then will

of the former Series.		of the latter Series.	
0. *n*	represent any Series of *Laterals* whose first Term is 0.	*n*. 0	the same Series inverted.
0. $p=$	the greatest *Trigon*	*p*. $p=$	the greatest *Trigon*
0. $q=$	the greatest *1st Pyram.*	*q*. $2q=$	the greatest *1st Pyram.*
0. $r=$	the greatest *2d Pyram.*	*r*. $3r=$	the greatest *2d Pyram.*
0. $s=$	the greatest *3d Pyram.*	*s*. $4s=$	the greatest *3d Pyram.*
0. $t=$	the greatest *4th Pyram.*	*t*. $5t=$	the greatest *4th Pyyam.*
&c.	*&c.*	*&c.*	*&c.*

Ex. gr.

I. Let the Number of Terms be two, or $n=1$, then will

of the former Series.	of the latter Series.
0. 1 represent the Series of *Laterals*.	1. 0 the same Series inverted.
0. 1= the greatest *Trigon*	1. 1=1×1 the greatest *Trigon*
0. 1= the greatest *1st Pyram.*	1. 2=2×1 the greatest *1st Pyram.*
0. 1= the greatest *2d Pyram.*	1. 3=3×1 the greatest *2d Pyram.*
0. 1= the greatest *3d Pyram.*	1. 4=4×1 the greatest *3d Pyram.*
0. 1= the greatest *4th Pyram.*	1. 5=5×1 the greatest *4th Pyramidal.*
&c. &c.	&c. &c.

II. Let the Number of Terms be three, or $n=2$, then will

of the former Series.	of the latter Series.
0. 1. 2 be the Series of *Laterals*.	2. 1. 0 the same Series inverted.
0. 1. 3= the greatest *Trigon*	2. 3. 3=1×3 the greatest *Trig.*
0. 1. 4= the greatest *1st Pyramidal*	2. 5. 8=2×4 the greatest *1st Pyramidal*
0. 1. 5= the greatest *2d Pyramidal*	2. 7. 15=3×5 the greatest *2d Pyramidal*
0. 1. 6= the greatest *3d Pyramidal*	2. 9. 24=4×6 the greatest *3d Pyramidal*
0. 1. 7= the greatest *4th Pyramidal*	2. 11. 35=5×7 the greatest *4th Pyramidal*
&c. &c.	&c. &c.

III. Let the Number of Terms be four, or $n=3$, then will

of the former Series.	of the latter Series.
0. 1. 2. 3 be the Series of *Laterals*.	3. 2. 1. 0 the same Series inverted.
0. 1. 3. 6= the greatest *Trigon*	3. 5. 6. 6=1×6 the greatest *Trigon.*
0. 1. 4. 10= the greatest *1st Pyramidal*	3. 8. 14. 20=2×10 the greatest *1st Pyr.*
0. 1. 5. 15= &c.	3. 11. 25. 45=3×15, the &c.
0. 1. 6. 21.	3. 14. 39. 84=4×21.
0. 1. 7. 28.	3. 17. 56. 140=5×28.
&c.	&c.

But

But this will always be ſo, if the Number of Terms be 5, 6, 7, 8, 9, 10, *&c. ad infinitum*; as is plain from a due Conſideration of the Nature and Structure of ſuch Numbers, which is here univerſally repreſented by the Letters p p, q $2q$, r $3r$, s $4s$, t $5t$, *&c.* as above. *Q. E. D.*

PROBLEM LXXII.

632. To ſum up a Series of *Trigons* with their 1ſt, 2d, 3d, 4th, *&c.* *Pyramidals*, whoſe Root or Number of Terms is n. (In. 229, 232.)

Effection.

Put p for the greateſt *Trigon*, q the greateſt 1ſt *Pyramidal*, r the greateſt 2d *Pyramidal*, s the greateſt 3d *Pyramidal*, t the greateſt 4th *Pyramidal*, *&c.* Then I ſay

$$p=n\times\frac{n+1}{2}=\frac{n+0}{1}\times\frac{n+1}{2} \qquad \text{(In. 213.)}$$

$$q=p\times\frac{n+2}{3}=\frac{n+0}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$$

$$r=q\times\frac{n+3}{4}=\frac{n+0}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$$

$$s=r\times\frac{n+4}{5}=\frac{n+0}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}$$

$$t=s\times\frac{n+5}{6}=\frac{n+0}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}\times\frac{n+5}{6} \qquad Q.\ E.\ E.$$

&c. &c. &c. &c. &c. &c. &c. &c. &c.

Demonſtration.

Ex. gr. Suppoſe $n=5$, then

I. The Series of *Laterals* may be repreſented

$$\left.\begin{array}{l}\text{by } 1.\ \ 1+1.\ 1+2.\ 1+3.\ 1+4=n \text{ or } \dfrac{n+0}{1}\\ \text{or by } n-4.\ n-3.\ n-2.\ n-1.\ n-0=n\end{array}\right\}$$

II. The *Triangular Numbers* formed of theſe *Laterals*, from the

$$\left\{\begin{array}{l}\text{former Series are } 1.\ \ 2+1.\ \ 3+3.\ \ 4+6.\ \ 5+10.\text{ or } n+10=p\\ \text{latter Series are } n-4.\ 2n-7.\ 3n-9.\ 4n-10.\ 5n-10.\text{ or } nn-10=p\end{array}\right\}$$

But

But the *Trigon* thus found in the first Value of p, (which is here 10,) will always be equal to that in the second Value of p, whatsoever n shall be, (In. 631.) *i. e.* $10=10$, then because $10=p-n$ from the first, and $10=nn-p$ from the second. Therefore

$$\begin{array}{r|l|l}
 & 1 & p-n=n^2-p \quad \text{(In. 21.)} \\
1+p & 2 & 2p-n=n^2 \\
2+n & 3 & 2p=n^2+n \\
3\div 2 & 4 & p=\dfrac{n^2+n}{2}=\dfrac{n+0}{1}\times\dfrac{n+1}{2}
\end{array}$$

the Teorem for summing up a a Series of *Laterals*: the same with that (In. 213.)

III. The *first Pyramidals* formed of those *Triangulars*: From the

$$\left\{\begin{array}{l}
\text{former Series are } 1. \quad 3+1. \quad 6+4. \quad 10+10. \quad 15+20=p+20=q \\
\text{latter Series are } n-4. \; 3n-11. \; 6n-20. \; 10n-30. \; 15n-40=pn-40=p
\end{array}\right\}$$

But the *first Pyramidal* thus found in the first Value of q, which is here 20, will always be subduple of that in the second Value of q, whatsoever n shall be, (In. 631.) *i. e.* $20=\frac{40}{2}$; then because $20=q-p$ from the first, and $40=pn-q$ from the second, therefore

$$\begin{array}{r|l|l}
 & 1 & 2q-2p=pn-q \\
1+q & 2 & 3q-2p=pn \\
2+2p & 3 & 3q=pn+2p \\
3\div 3 & 4 & q=\dfrac{pn+2p}{3}=p\times\dfrac{n+2}{3} \text{ or } \dfrac{n+0}{1}\times\dfrac{n+1}{2}\times\dfrac{n+2}{3}
\end{array}$$

the Theorem for summing up a Series of *Triangulars*.

IV. The *second Pyramidals* formed of the *first Pyramidals*, from the

$$\left\{\begin{array}{l}
\text{former Series are } 1. \quad 4+1. \quad 10+5. \quad 20+15 \quad 35+35=q+35=r \\
\text{latter Series are } n-4. \; 4n-15. \; 10n-35. \; 20n-65. \; 35n-105=qn-105=r
\end{array}\right\}$$

But the *second Pyramidal* thus found in the first Value of r, (which is here 35,) will always be subtriple of that in the second Value of r, whatsoever n shall be, (In. 631.) *i. e.* $35=\frac{105}{3}$; Then because $35=r-q$ from the first, and $105=qn-r$ from the second, therefore

$$\begin{array}{r|l|l}
 & 1 & 3r-3q=qn-r \\
1+r & 2 & 4r-3q=qn \\
2+3q & 3 & 4r=qn+3q \\
3\div 4 & 4 & r=\dfrac{qn+3q}{4}=q\times\dfrac{n+3}{4} \text{ or } \dfrac{n+0}{1}\times\dfrac{n+1}{2}\times\dfrac{n+2}{3}\times\dfrac{n+3}{4}
\end{array}$$

the

Theorem

Theorem for summing up any Rank of first *Pyramidals Triangular*. And by the same Law, it is plain, will proceed the Sums of every Series of higher *Pyramidals*, whatsoever the Number of Terms shall be (In. 631.) *viz.* $\frac{n+0}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} \times \frac{n+5}{6} \times \frac{n+6}{7} \times \frac{n+7}{8}$ *&c.*

Corollary IX.

633. Hence we learn to form the *Unciæ* of Powers raised from a Binomial Root. For it has been shewn that the *Unciæ* of every first Term is Unity, of every second Term the Exponent of the Power m, of the third Term a Triangular Number whose Root is $m-1$, of the fourth Term a first Pyramidal Triangular whose Root is $m-2$, of the fifth Term a second Pyramidal whose Root is $m-3$, of the sixth Term a third Pyramidal whose Root is $m-4$, *&c.* (In. 409.) But the Trigon whose Root is $m-1$ is $m \times \frac{m-1}{2}$, the first Pyramidal whose Root is $m-2$ is $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, the second Pyramidal whose Root is $m-3$ is $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, the third Pyramidal whose Root is $m-5$ is $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ (In. 632.) Whence have we the Theorem (In. 464.) for finding the *Unciæ* of Powers, *viz.* $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} \times \frac{m-6}{7} \times \frac{m-7}{8} \times \frac{m-8}{9}$ *&c.*

Corollary X.

634. Hence in like manner have we Theorems for determining all the simple Combinations of any Number of Things. For if n be put for the Number of Things to be combined, the Number of all the Combinations by *Pairs* will be $n \times \frac{n-1}{2}$, by *Ternaries* $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, by *Quaternaries* $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, *&c.* (In. 236.)

Corollary XI.

635. Therefore the Number of all the possible *Simple Combinations* of any Number of Things n, is the same with the Sum of all the *Unciæ* of the n Power of any Binomial $b \pm e$, excepting the *Unciæ* of the two first Terms

(*i. e.* excepting $1+n$) But fuch a Sum is $= 2^n - n - 1$ (In. 410.) Therefore $2^n - n - 1$ equals the Number of all the poffible *Simple Combinations* of any Number of Things n.

Ex. gr. Let $n=8$, then all the *Simple Combinations* by *Pairs* will be $n \times \frac{n-1}{2} = 28$, by *Ternaries* $n \times \frac{n-1}{2} \times \frac{n-2}{3} = 56$, by *Quaternaries* $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} = 70$, by *Fives* $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} = 56$, by *Sixes* $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} = 28$, by *Sevens* $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{n-6}{7} = 8$, by *Eights* $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{n-6}{7} \times \frac{n-7}{8} = 1$. In all $28+56+70+56+28+8+1=247=2^8-8-1$.

Problem LXXIII.

636. To fum up any Series of *Polygons*, with their *Firft*, *Second*, *Third*, *&c. Pyramidals*, whofe Root or Number of Terms is n, common Difference of the *Arithmetical Series* whence they are formed d, and greateft Term of the fame G (In. 229.)

Effection.

Put P for the greateft *Polygon* or Sum of the Arithmetical Series (which will be either *Trigon*, *Tetragon*, *Pentagon*, *Hexagon*, *&c.* according as d is 1, 2, 3, 4, *&c.*) Q for the greateft *firft Pyramidal* of the fame Denomination, or Sum of the Series of *Polygons*; R for the greateft *fecond Pyramidal*, or Sum of the Series of *Firft Pyramidals*; S for the greateft *third Pyramidal*, or Sum of the Series of *Second Pyramidals*; T for the greateft *fourth Pyramidal*, *&c.* Alfo let p, q, r, s, t, *&c.* reprefent the fame here as in the laft Problem, which (if $d=1$ and confequently $G=n$) muft be the fame with P, Q, R, S, T, *&c.* and therefore the Effection here would be the fame as in the laft; but if $d=$ any other Number befides Unity, and confequently G be different from n: Then I fay

$P = n\times$

$$P=n\times\frac{G+1}{2}=\frac{nG+n}{2} \qquad \text{(In. 212.)}$$

$$Q=p\times\frac{G+2}{3}=n\times\frac{n+1}{2}\times\frac{G+2}{3}$$

$$R=q\times\frac{G+3}{4}=n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{G+3}{4}$$

$$S=r\times\frac{G+4}{5}=n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+4}{4}\times\frac{G+4}{5}$$

$$T=s\times\frac{G+5}{6}=n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+4}{4}\times\frac{n+4}{5}\times\frac{G+5}{6}$$

&c. &c. &c. *Q. E. E.*

Demonſtration.

Ex. gr. Suppoſe $n=5$ as before, then

I. The *Arithmetical Series* whoſe firſt Term is Unity, and common Difference d.

$$\left\{\begin{array}{llllll} \text{is} & 1. & 1+d. & 1+2d. & 1+3d. & 1+4d=G \\ \text{or } G-4d. & G-3d. & G-2d. & G-d. & G-0 & =1+nd-d \end{array}\right\}$$

II. The *Polygons* formed from the

$$\left\{\begin{array}{llllll} \text{former Series are} & 1. & 2+d. & 3+3d. & 4+6d. & 5+10d=n+10d=P \\ \text{latter Series are} & G-4d. & 2G-7d. & 3G-9d. & 4G-10d. & 5G-10d=nG-10d=P \end{array}\right\}$$

But the *Polygon* thus found in the former Value of P, which is here $10d$, will always be equal to that in the latter Value of P, whatſoever n ſhall be, (In. 631.) *i. e.* $10d=10d$; then becauſe $10d=P-n$ from the former, and $10d=nG-P$ from the latter: Therefore

$$\begin{array}{r|l|l} & 1 & P-n=nG-P \\ 1+P & 2 & 2P-n=nG \\ 2+n & 3 & 2P=nG+n \\ 3\div 2 & 4 & P=\dfrac{nG+n}{2}=n\times\dfrac{G+1}{2} \end{array}$$

the Theorem for ſumming up any Rank of *Arithmetical Progreſſionals*, whoſe firſt Term $l=1$.

III. The

III. The *first Pyramidals* from those *Polygons*, from the

$$\left\{\begin{array}{llllll} \text{former Series are} & 1. & 3+d. & 6+4d. & 10+10d. & 15+20d=p \\ & & & & & +20d=Q \\ \text{latter Series are} & G-4d. & 3G-11d. & 6G-20d. & 10G-30d. & 15G-40d=pG \\ & & & & & -40d=Q \end{array}\right\}$$

But the *first Pyramidal* thus formed in the former Value of Q, which is here $20d$, will always be subduple of that in the latter Value of Q (In. 631.) *i. e.* $20d=\frac{40d}{2}$, then because $20d=Q-p$ from the former, and $40d=pG-Q$ from the latter; therefore

$$\begin{array}{r|l|l} & 1 & 2Q-2p=pG-Q \\ 1+Q & 2 & 3Q-2p=pG \\ 2+2p & 3 & 3Q=pG+2p \\ 3\div 3 & 4 & Q=\frac{pG+2p}{3}=p\times\frac{G+2}{3} \end{array}$$

, or by substituting the Value of p (In. 631.) $Q=n\times\frac{n+1}{2}\times\frac{G+2}{3}$ the Theorem for summing up any Rank of *Polygons*.

IV. The *second Pyramidals* formed from those first Pyramidals, from the

$$\left\{\begin{array}{llllll} \text{former Series are} & 1. & 4+d. & 10+5d. & 20+15d. & 35+35d. \\ & & & & & =q\ \ +35d=R \\ \text{latter Series are} & G-4d. & 4G-15d. & 10G-35d. & 20G-65d. & 35G-105d \\ & & & & & =qG-105=R \end{array}\right\}$$

But the *First Pyramidal* thus formed in the former Value of R, which is here $35d$, will always be subtriple of that in the latter Value of R, *i. e.* $35d=\frac{105d}{3}$, then because $35d=R-q$ from the former, and $105d=qG-R$ from the latter; therefore

$$\begin{array}{r|l|l} & 1 & 3R-3q=qG-R \\ 1+R & 2 & 4R-3q=qG \\ 2+3q & 3 & 4R=qG+3q \\ 3\div 4 & 4 & R=\frac{qG+3q}{4}=q\times\frac{G+3}{4} \end{array}$$

or by substituting the Value of q (In. 632.) $R=n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{G+3}{4}$ the Theorem for summing up any Rank of first *Pyramidals* of what Denomination soever. And, by the same

Law

Law will proceed the Sums of every Rank of higher *Pyramidals*, whatsoever the Number of Terms n shall be; as in the Effection above. *Q. E. D.*

Corollary XII.

637. And because $G=1+nd-d$, in the foregoing *Arithmetical Series*, whose least Term is 1, Number of Terms n, and common Difference d, (In. 207.) therefore substituting this Value of G in this Theorem, $Q=n\times\frac{n+1}{2}\times\frac{G+2}{3}=\frac{G\times\overline{nn+n}+2\times\overline{nn+n}}{6}$, and we shall have the following universal Theorem for summing up any Rank of *Polygons*, from n and d given, $Q=\frac{n^3d+3nn+3n-dn}{6}$. Whence are deduced the following particular Theorems, *viz.*

If the *Polygons* to be summed up be a Rank of

Laterals		0	$\frac{3n^2+3n}{6}=\frac{nn+n}{2}$
Trigons		1	$\frac{n^3+3n^2+n}{6}$
Tetragons		2	$\frac{2n^3+3n^2+n}{6}$
Pentagons	*i. e.* if $d=$	3	the Sum Q will be $\frac{n^3+n^2}{2}$
Hexagons		4	$\frac{4n^3+3n^2-n}{6}$
Heptagons		5	$\frac{5n^3+2n^2-2n}{6}$
Octogons		6	$\frac{2n^3+n^2-n}{2}$
&c.		&c.	&c.

Corollary XIII.

638. And all the Q's thus found are a Rank of Terms in $\div$, whose least Term is $\frac{nn+n}{2}$, Number of Terms $d+1$, and common Difference $\frac{nnn-n}{6}$; therefore if for the greatest Term be put Q, the Sum of all the Terms will be

Z $dnn+dn$

$\frac{dnn+dn+nn+n+2dQ+2Q}{4}$ (Theo. V. In. 599.) *i. e.* if the higheſt Term Q be a.

$$\left.\begin{array}{l} \textit{Lateral} \\ \textit{Trigon} \\ \textit{Tetragon} \\ \textit{Pentagon} \\ \textit{Hexagon} \\ \textit{Heptagon} \\ \textit{Octogon} \end{array}\right\} \text{the Sum will be} \left\{\begin{array}{l} \frac{nn+n+2Q}{4}=\frac{nn+n}{2} \\ \frac{nn+n+2Q}{2}=\frac{n^3+6n^2+5n}{6} \\ \frac{3nn+3n+6Q}{4}=\frac{n^3+6n^2+3n}{3} \\ nn+n+2Q=n^3+2n^2+n \\ \frac{5nn+5n+10Q}{4}=\frac{10n^3+15n^2+5n}{6} \\ \frac{3nn+3n+6Q}{2}=\frac{5n^3+6n^2+n}{3} \\ \frac{7nn+7n+14Q}{4}=\frac{7n^3+7n^2}{2} \end{array}\right.$$

&c. *&c.* *&c.*

LEMMA II. to the following Problems.

639. If n be put indifferently for any Integer, then the Scale of Powers whoſe Root is n, will be n^0, n^1, n^2, n^3, n^4, n^5, *&c.* Alſo the Scale of Powers whoſe Root is $n+1$ will be $\overline{n+1}^0$, $\overline{n+1}^1$, $\overline{n+1}^2$, $\overline{n+1}^3$, $\overline{n+1}^4$, $\overline{n+1}^5$: And if, for the Sum of all the

$$\left.\begin{array}{l} \textit{Laterals} \\ \textit{Squares} \\ \textit{Cubes} \\ \textit{Biquadrates} \\ \textit{Fifth Powers} \end{array}\right\} \begin{array}{l} \text{whoſe Roots are 1,} \\ \text{2, 3, 4, 5, \&c. to} \\ n\text{, be put} \end{array} \left\{\begin{array}{l} A=\frac{nn+n}{2}\ \text{(In. 213.)} \\ B=\frac{2n^3+3n^2+n}{6}\ \text{(In. 636.)} \\ C \\ D \\ E \end{array}\right.$$

&c. *&c.*

then I ſay, whatſoever n be, that $B+\overline{1+n}^2$, $C+\overline{1+n}^3$, $D+\overline{1+n}^4$, *&c.* will be equal to the Sum of the Series of *Squares*, *Cubes*, *Biquadrates*, *&c.* reſpectively, whoſe Number of Terms is $1+n$. The Aſſertion is ſelf-evident to any who underſtands what he reads; nevertheleſs, if need ſo require, the Demonſtration thereof may be caſt into the following Syllogiſm.

The

The Sum of a Series of Homologous Powers, whose *Roots* are 1, 2, 3, 4, 5, &c. to n, is equal to B, C, D, &c. according as the Powers are *Squares*, *Cubes*, *Biquadrates*, &c. by Hypothesis.

But the Sum of a Series of Homologous Powers whose *Roots* are 1, 2, 3, 4, 5, &c. to $1+n$ is more than the foregoing Sums by $\overline{1+n}^2$, $\overline{1+n}^3$, $\overline{1+n}^4$, &c. according as the Powers are *Squares*, *Cubes*, *Biquadrates*, &c. because $\overline{1+n}^1$ is the next Integer above n: Therefore, &c.

Lemma III.

640. Again put m universally for the Exponent of any Series of Homologous Powers to be summed up, whose *Roots* are in $\div$, begining with Unity, whose Number of Terms are $\overline{1+n}$, and the common Difference of the Arithmetical Progression is d. Then I say the Sum of the Series of *Squares* will be $1+n+mdA+m\times\frac{m-1}{2}ddB$, or $1+n+2dA+ddB$: The Sum of the Series of *Cubes* will be $1+n+mdA+m\times\frac{m-1}{2}ddB+m\times\frac{m-1}{2}\times\frac{m-2}{3}dddC$, or $1+n+3dA+3ddB+dddC$: The Sum of the Series of *Biquadrates* will be $1+n+mdA+m\times\frac{m-1}{2}ddB+m\times\frac{m-1}{2}\times\frac{m-2}{3}dddC+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$ $ddddD$, or $1+n+4dA+6ddB+4dddC+ddddD$: The Sum of the Series of *Fifth Powers* will be $1+n+mdA+m\times\frac{m-1}{2}ddB+m\times\frac{m-1}{2}\times\frac{m-2}{3}dddC+m$ $\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}ddddD+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}dddddE$, or $1+n$ $+5dA+10ddB+10dddC+5ddddD+dddddE$, &c. proceeding according to ln. 632. Note, A, B, C, D, &c. are here the same as in the last.

Demonstration by Induction.

Suppose *Ex. gr.* $n=4$. Then.

I. The *Arithmetical Series*, whose first Term is Unity, Common Difference is d, and Number of Terms is $1+n$, is

$$1.\ 1+d.\ 1+2d.\ 1+3d.\ 1+4d.=1+nd.$$

Whose Sums are

$$1.\ 1+1+d.\ 1+2+3d.\ 1+3+6d.\ 1+4+10d=1+n+.Ad=\frac{2+2n+nnd+nd}{2},$$

reassuming the Value of A as found (ln. 213.)

II. The

II. The Rank of *Squares* of the *Arithmetical Series* is

$1.\ 1+2d\times1+dd.\ 1+2d\times2+4dd.\ 1+2d\times3+9dd.\ 1+2d\times4+16dd=\overline{1+nd}^2$ or $1+2nd+nndd$.

Whose Sums are

$1.\ 1+1+2d\times1+d^2.\ 1+2+2d\times3+5d^2.\ 1+3+2d\times6+14dd.\ 1+4+2d\times10+30d^2=1+n+2dA+ddB=\frac{6+6n+6d\times\overline{nn+n}+dd\times\overline{2n^3+3n^2+n}}{6}$ (In. 213, 636.)

III. The Rank of *Cubes* are

$1.\ 1+3d\times1+3dd\times1+d^3.\ 1+3d\times2+3dd\times4+8d^3.\ 1+3d\times3+3dd\times9+27d^3.\ 1+3d\times4+3dd\times16+64d^3=\overline{1+nd}^3$.

Whose Sums are

$1.\ 1+1+3d\times1+3dd\times1+d^3.\ 1+2+3d\times3+3dd\times5+9d3^3.\ 1+3+3d\times6+3dd\times14+36d^3.\ 1+4+3d\times10+3dd\times30+100d^3=1+n+3dA+3ddB+dddC$.

IV. The Rank of *Biquadrates* are

$1.\ 1+4d\times1+6dd\times1+4d^3\times1+d^4.\ 1+4d\times2+6dd\times4+4d^3\times8+16d^4.\ 1+4d\times3+6dd\times9+4d^3\times27+81d^4.\ 1+4d\times4+6dd\times16+4d^3\times64+256d^4=\overline{1+nd}^4$.

Whose Sums are

$1.\ 1+1+4d\times1+6d^2\times1+4d^3\times1+d^4.\ 1+2+4d\times3+6d^2\times5+4d^3\times9+17d^4.\ 1+3+4d\times6+6dd\times14+4d^3\times63+98d^4.\ 1+4+4d\times10+6dd\times30+4d^3\times100+354d^4=1+n+4dA+6ddB+4dddC+ddddD$.

And thus it is plain, it will always be, whatever be the Value of n, and how high soever the Denomination of the Powers. So that if z be put to represent the Sum of any Series of m Powers, whose Roots are the *Laterals* 1, 2, 3, 4, 5, 6, *&c.* and whose Number of Terms from Unity is n, Y the Sum of the Series of $\overline{m-1}$ Powers; X the Sum of the Series of $\overline{m-2}$ Powers; W the Sum of the Series of $\overline{m-3}$ Powers; V the Series of $\overline{m-4}$ Powers; T the Series of $\overline{m-5}$ Powers, *&c.* And if for the *Unciæ* be put $m, p, q, r, s,$ *&c.* (In. 404.) then will the Sum of any Series of m Powers, whose Roots are in $\div$, beginning with Unity (the Common Difference being d, and Number of Terms $n+1$) be equal to $Zd^m+mYd^{m-1}+pXd^{m-2}+qWd^{m-3}+rVd^{m-4}+sTd^{m-5}+$ *&c.* $+mAd+n+1$. *Q. E. D.*

COROL-

COROLLARY XIV.

641. And because in the Series of *Laterals* 1, 2, 3, 4, 5, *&c.* $d=1$, therefore the Sum of the *Squares* of such a Series (whose Number of Terms, and consequently last Term is $1+n$) equals $1+n+2A+B$: Of *Cubes* equals $1+n+3A+3B+C$: Of *Biquadrates* equals $1+n+4A+6B+4C+D$: Of *Fifth Powers* equals $1+n+5A+10B+10C+5D+E$: *&c.* And universally the Sum of all the m Powers of such a Series equals $Z+mY+pX+qW+rV+sT$ *&c.* $+mA+n+1$, or otherwise $1+n+mA+pB+qC+rD+sE+$ *&c.* $+mY+Z$.

COROLLARY XV.

642. It also appears from the last Lemma, in summing up the Series of *Cubes*, whose Roots are 1, 2, 3, 4, *&c.* to n the last Term, that C is always equal to AA.

PROBLEM LXXIV.

643. To determine the Sum of any Series of Powers in Numbers, whose Roots are 1, 2, 3, 4, 5, *&c.* the Number of Terms, and consequently the last Term, being represented by n.

Effection.

I. To find the Sum of the Rank of *Squares* B.

Because the Sum of the Series of *Cubes*, whose Roots are 1, 2, 3, 4, 5, *&c.* and Number of Terms is $\overline{1+n}$, has been shewn to be $1+n+3A+3B+C$ (In. 640.) and also to equal $C+\overline{1+n}^3$ or $C+n^3+3n^2+3n+1$ (In. 638.) Therefore

	1	$C+3B+3A+n+1=C+n^3+3n^2+3n+1$
$1-C-n-1$	2	$3B+3A=n^3+3n^2+2n$
$2-3A$	3	$3B=n^3+3n^2+2n-3A$
	4	$A=\dfrac{nn+n}{2}$ (In. 213.)
$\therefore$	5	$3B=n^3+3n^2+2n-3\times\dfrac{nn+1}{2}$
Whence	6	$B=\dfrac{2n^3+3n^2+n}{6}$ the Theorem for summing up

a Series of *Squares* or *Tetagrons* (In. 636.)

II. To find the Sum of the Series of *Cubes* C.

Becauſe the Sum of the Series of *Biquadrates* whoſe Number of Terms is $1+n$, is equal to $1+n+4A+6B+4C+D$ (In. 640.) and to $D+\overline{n+1}^4$ (In. 638.) therefore

$$\begin{array}{r|l|l}
 & 1 & D+4C+6B+4A+n+1=D+n^4+4n^3+6n^2+4n+1 \\
1-D-n-1 & 2 & 4C+6B+4A=n^4+4n^3+6n^2+3n \\
 & 3 & A=\dfrac{nn+n}{2}\text{, and } B=\dfrac{2n^3+3n^2+n}{6}\text{ as above} \\
\therefore & 4 & 4C+2n^3+3n^2+n+2\times\overline{nn+n}=n^4+4n^3+6n^2+3n \\
\text{Whence} & 5 & C=\dfrac{n^4+2n^3+n^2}{4}=AA
\end{array}$$

(In. 641.) The Theorem for ſumming the Series of *Cubes*.

III. To find the Sum of the Series of *Biquadrates* D.

Becauſe the Sum of the Series of *Fifths Powers*, whoſe Number of Terms from Unity is $1+n$, is equal to $1+n+5A+10B+10C+5D+E$ (In. 640.) and to $E+\overline{n+1}^5$ (In. 638.) Therefore $E+5D+10C+10B+5A+n+1 =E+\overline{n+1}^5$; whence proceeding as before, by ſubſtituting the Values of A, B, and C already found, there will come out $D=\dfrac{6n^5+15n^4+10n^3-n}{30}$ the Theorem for ſumming up the Series of *Biquadrates*.

In like Manner, by a due Management of the Equation $F+6E+15D+20C+15B+6A+n+1=F+\overline{n+1}^6$, ſubſtituting the Values of A, B, C, and D, there will come out $E=\dfrac{2n^6+6n^5+5n^4-n^2}{12}$ the Theorem for ſumming up the Series of *Fifth Powers*, whoſe Number of Terms from Unity is n.

After the ſame Manner the Learner may proceed, at his own Leiſure, to find Theorems for ſumming up the Series of *Sixth*, *Seventh*, *Eighth*, *Ninth*, *&c.* *Powers*.

And univerſally from the Equation $z+mY+pX+qW+rV+sT$, &c. $+mA+n+1=Z+\overline{n+1}^m$ (In. 640, and 638.) will be found $Y=\dfrac{\overline{n+1}^m}{m}-\dfrac{p}{m}X-\dfrac{q}{m}W-\dfrac{r}{m}V-\dfrac{s}{m}T$ &c. $-\dfrac{p}{m}A-\dfrac{1}{m}n-\dfrac{1}{m}$ the Theorem for ſumming up any

Series

Series of Powers whose Exponent is $\overline{n-1}$, and Number of Terms from Unity n.

Problem LXXV.

644. To determine the Sum of any Series of Powers in Numbers whose Roots are in $\div$, the Number of Terms from Unity being $1+n$, and the common Difference d.

Effection.

I. To find the Sum of the Rank of *Squares.*

Because the Sum of the Series of *Squares* (whose Roots are in $\div$, and Number of Terms from Unity is $1+n$) has been shewn to be equal to $1+n+2dA+ddB$ (In. 640.) therefore substituting the Values of A and B as found (In. 642.) it will become $1+\begin{matrix}1\\d\\\frac{1}{6}dd\end{matrix}\,n^2+\begin{matrix}d\\\frac{1}{2}dd\end{matrix}\,nn+\frac{1}{3}ddn^3$ the Theorem for Summing the Series of *Squares.*

II. To find the Sum of the Rank of *Cubes.*

The Sum of the Series (whose Roots and Number of Terms are the same as above) is $1+n+3dA+3ddB+d^3C$ (In. 640.) therefore substituting the Values of A, B and C as found (In. 642.) you will have

$1+\begin{matrix}1\\\frac{3}{2}d\\\frac{1}{2}dd\end{matrix}\,n+\begin{matrix}\frac{3}{2}d\\\frac{3}{2}dd\\\frac{3}{4}ddd\end{matrix}\,n^2+\begin{matrix}dd\\\frac{1}{2}ddd\end{matrix}\,n^3+\frac{1}{4}dddn^4$ the Theorem for summing up the Series of *Cubes*, whose Number of Terms from Unity is $1+n$. And thus I suppose it will be easy for the Learner from (In. 640, and 642.) to raise Theorems for summing up any Series of higher Powers at Pleasure, without proceeding further.

Scholium VIII.

645. *Ex. gr.* Suppose a Rank of *Cubes* were to be summed up, whose Roots are 1, 2, 3, 4, 5, 6, &c. beginning from Unity, the Number of Terms being $60=n$, the Answer will be $C=\frac{n^4+2n^3+n^2}{4}=3348900$ the Sum required (In. 644.)

Again, suppose a Rank of *Squares*, whose Roots are in Arithmetical Progression, beginning from Unity; the Number of Terms being $101=n+1$, and common Difference $3=d$, then $B=1+\begin{matrix}1\\d\\\frac{1}{6}dd\end{matrix}\,n+\begin{matrix}d\\\frac{1}{2}dd\end{matrix}\,n^2+\frac{1}{3}ddn^3=$ 9165851 (In. 644.)

CHAP.

CHAP V.

Of the Arithmetick of Infinites.

Demonstration.

646. AN *Infinitessimal* is that which is conceived to be infinitely less than any assignable Quantity, as a Grain of Sand in Comparison with the whole Globe of the Earth; a Moment of Time in respect of a Million of Ages, *&c.*

COROLLARY XVI.

647. Every *Infinitessimal* is comparatively as nothing in respect of the Quantity to which it is an *Infinitessimal.*

COROLLARY XVII.

648. Hence whatever Quantities do only differ by an *Infinitessimal* are to be looked upon as equal, and consequently may be substituted one for another.

COROLLARY XVIII.

649. Hence also every inferior Power of an infinite Quantity (if I may be allowed the Expression) is an *Infinitessimal* in respect of its superior one, as being infinitely less than the other; *i. e.* the Square of an *Infinite* is an *Infinitessimal* in respect of the Cube, the Cube in respect of the Biquadrate, *&c.* therefore every inferior Power is to be looked upon as nothing in respect of its superior one.

COROLLARY XIX.

650. Whence, if a Series of Laterals 1, 2, 3, 4, 5, *&c.* be continued *ad Infinitum*; so that n the greatest Term and Number of Terms be infinite; then, rejecting all the *Infinitessimals* by the last, the Theorems $A=\frac{nn+n}{2}$ (In. 213.) will become $A=\frac{n^2}{2}$; $B=\frac{2n^3+3n^2+n}{6}$ will become $B=\frac{2n^3}{6}$ or $\frac{n^3}{3}$, $C=\frac{n^4+2n^3+n^2}{4}$ will become $\frac{n^4}{4}$; $D=\frac{6n^5+15n^4+10n^3-n}{30}$ will become $\frac{6n^5}{30}$ or $\frac{n^5}{5}$; $E=\frac{2n^6+6n^5+5n^4-n^2}{12}$ will become $\frac{2n^6}{12}$ or $\frac{n^6}{6}$; and universally the

Theorem

Theorem $Y=\frac{\overline{n+1}^m}{m}-\frac{p}{m}X-\frac{q}{m}W-\frac{r}{m}V-\frac{s}{m}T$ &c. $-\frac{p}{m}A-\frac{1}{m}n-\frac{1}{m}\times 1$ will become $Y=\frac{\overline{n+1}^m}{m}$ or $Y=\frac{n^m+mn^{m-1}+pn^{m-2}+qn^{m-3} \text{ \&c.}}{m}$ which by again rejecting all the *Infinitessimals* will become $Y=\frac{n^m}{m}$.

Corollary XX.

651. Whence lastly have we this general Theorem, That the Sum of every Series of Homologous Powers, whose common Exponent is m, and whose Roots are the natural Numbers 1, 2, 3, 4, 5, *&c. ad infinitum*, is to as many equal to the greatest as 1 to $m+1$. In particular, If $m=1$, the Sum of the Series of *First Powers* or *Roots A* is to as many equal to the greatest, as 1 to 2: If $m=2$, the Sum of the Series of *Squares B* is to as many equal to the greatest as 1 to 3: If $m=3$, the Sum of the Series of *Cubes C* is to as many equal to the greatest as 1 to 4. If $m=5$, the Sum of the Series of *Biquadrates D* is to as many equal to the greatest as 1 to 5, *&c. ad infinitum.*

Scholium IX.

652. And by the same Manner many other Theorems may be raised from In. 632, 635 and 641.

The Inventor of the Arithmetick of Infinites was the great Dr. *Wallis*, *Savilian Professor* of *Geometry* in *Oxford*; the use of which is generally in the Business of Geometry. But since the admirable Invention of Fluxions, it is for the most Part laid aside.

The End of the Fifth PART.

ARITHMETICAL INSTITUTIONS.

PART VI.

The APPLICATION of SPECIES ALGORISM to the EFFECTION of INDETERMINATE PROBLEMS.

CHAP. I.

Of Single LATERAL EQUALITIES.

PROBLEM I.

653. TO find two Integers a, e, whose Sum added to their Product is equal to a given Integer n.

Effection.

Whence
1 | $ae+a+e=n$ by the Question.
2 | $a=\frac{n-e}{e+1}=\frac{-e+n}{e+1}=-1+\frac{n+1}{e+1}$ (In. 401.)

Therefore for e must be assumed some Number, which being added to Unity will divide $n+1$ without a Remainder. *Ex. gr.* If $n=19$, then e may

may $=3$, according to which $a=\frac{n+1}{e+1}-1=4$: Or e may $=4$, according to which $a=3$; or e may $=9$, according to which $a=1$.

Problem II.

654. To find two Integers a, e, whose Difference added to their Product is equal to a given Number n.

Effection.

$$\text{Whence}\left|\begin{array}{l|l} 1 & ae+a-e=n, \text{ by the Question.} \\ 2 & a=\frac{e+n}{e+1}=1+\frac{n+1}{e+1} \end{array}\right.$$

Therefore for e must be assumed some Number which being added to Unity will divide $n+1$ without a Remainder. *Ex. gr.* If $n=28$, then e may $=2$, according to which $a=\frac{n+1}{e+1}+1=10$, or e may $=8$, according to which $a=4$.

Scholium I.

55. In the Effection then of the two foregoing Problems, it is plain that $n+1$ in the former, and $n-1$ in the latter must always be composit Numbers; and for $e+1$ may be assumed any of their Aliquot Parts.

Problem III.

656. To find two Integers a, e, whose Sum is equal to their Product less the latter.

Effection.

$$\text{Whence}\left|\begin{array}{l|l} 1 & ae-e=a+e \text{ by the Question.} \\ 2 & a=\frac{2e}{e-1}=2+\frac{2}{e-1} \end{array}\right.$$

Therefore for e here can be assumed no Integer but 2, and accordingly $a=\frac{2e}{e-1}=4$.

Problem IV.

657. To find two Numbers, a the greater, e the lesser, whose Sum is equal to the Difference of their Squares.

Effection.

Effection.

$$\begin{array}{r|l|l} & 1 & aa-ee=a+e \text{ by the Question.} \\ 1\div\overline{a+e} & 2 & a-e=1 \end{array}$$

Therefore any two Numbers will anſwer the Queſtion whoſe Difference is Unity. *Ex. gr.* If $a=2$, then $e=1$. If $a=3$, $e=2$. If $a=4$, $e=3$. &c.

Problem V.

658. To find two Numbers, a the greater, e the leſſer, whoſe Difference is equal to the Difference of their Squares.

Effection.

$$\begin{array}{r|l|l} & 1 & a^2-e^2=a-e \text{ by the Question.} \\ 1\div\overline{a-e} & 2 & a+e=1 \end{array}$$

Therefore the Numbers ſought may be any two Fractions whoſe Sum is Unity. *Ex. gr.* If $a=0.9$, $e=0.1$. If $a=0.8$, $e=0.2$. If $a=0.7$, $e=0.3$, &c.

Problem VI.

659. To find two Numbers, a the greater, e the leſſer, the Sum of whoſe Squares is equal to the Difference between their Product and the Difference of their Cubes.

Effection.

$$\begin{array}{r|l|l} & 1 & a^3-e^3-ae=a^2+e^2 \text{ by the Question.} \\ 1+ae & 2 & a^3-e^3=a^2+ae+ee \\ 2\div\overline{aa+ae+ee} & 3 & a-e=1 \end{array}$$

Therefore any two Numbers, whoſe Difference is Unity, will ſatisfy the Queſtion; as in Prob. IV.

Scholium II.

660. The three laſt Problems are higher than Lateral Equalities, but are reduced to ſuch in their Effection.

Problem VII.

661. To find three Integers a, e, y, whoſe Sum is equal to the Product of the firſt and ſecond, leſs the Product of the ſecond and third.

Effection.

$$\begin{array}{r|l|l} & 1 & a+e+y=ae-ey \text{ by the Question.} \\ \text{Whence} & 2 & a=\dfrac{ey+e+y}{e-1}=y+1+\dfrac{2y+1}{e-1} \end{array}$$

Therefore for e and y may be assumed any two Integers, so that $2y+1$ be divisible be $e-1$. *Ex. gr.* If $y=2$ and $e=6$, then $a=y+1+\frac{2y+1}{e-1}=4$. If $y=3$ and $e=8$, then $a=5$. If $y=4$, and $e=10$, then $a=6$. &c.

Problem VIII.

662. To find four Integers a, e, y, u, whose Sum is equal to the Product of the first and fourth.

Effection.

$$\text{Whence}\left|\begin{array}{l|l} 1 & au=a+e+y+u \text{ by the Question.} \\ 2 & a=\frac{u+y+e}{u-1}=1+\frac{y+e+1}{u-1} \end{array}\right.$$

Therefore for e, y, u, may be assumed any three Integers, so that $y+e+1$ be divisible by $u-1$. *Ex. gr.* If $y=11$, and $e=15$, or $y+e=26$, then must $u=4$, and accordingly $a=1+\frac{y+e+1}{u-1}=10$; or $u=10$ and $a=4$. If $y+e=35$, then may $u=4$, and accordingly $a=13$; or $u=5$, and $a=10$; or $u=7$, and $a=7$; or $u=10$, and $a=5$; or $u=19$, and $a=3$.

Scholium III.

663. Hence it is plain, in the Effection of all Questions of this Sort, that Numbers must be assumed for as many unknown Quantities as there are more than the Number of given Equations.

Problem. IX.

664. To find what particular Quantities of two or more Ingredients (whose Prices are given) will compose a Mixture that may be sold at a given mean Price.

Example 1. How much Wheat at 5 *s.* 4 *d.* (or 64 *d.*) the Bushel, and Rye at 3 *s.* 7 *d.* (or 43 *d.*) the Bushel will compose a Mixture that may be sold at 4 *s.* 7 *d.* or 55 *d.*) the Bushel.

For the Quantity of Wheat put a, and Rye e.

$$\begin{array}{r|l|l} \text{Then} & 1 & 64a+43e=55a+55e \text{ by the Question.} \\ & 2 & 55a+55e=55a+55e \\ 1-2 & 3 & 9a-12e=0, \text{ or } 9a=12e \\ 3\div 3 & 4 & 3a=4e \end{array}$$

Therefore

Therefore any two Numbers may be aſſumed for a and e, whereof $3a=4e$; or whereof a is to e as 4 to 3: As in the following Table.

a	e
4	3
8	6
12	9
16	12
&c.	&c.

ad infinitum.

Example 2. How much Wine of each of the following ſorts, *viz.* of 16 *d.* 10 *d.* 8 *d.* and 6 *d.* the Quart, will compoſe a Mixture that may be ſold for 9 *d.* the Quart;

Put a for the Number of Quarts of that at 16 *d.* e at 10 *d.* y at 8 *d.* and u at 6 *d.*

Then	1	$16a+10e+8y+6u=9a+9e+9y+9u$ by the Queſtion.
	2	$9a+9e+9y+9u=9a+9e+9y+9u$
1—2	3	$7a+e-y-3u=0$ or $7a=y+3u-e$
3÷5	4	$a=\frac{y+3u-e}{7}$

Therefore any four Numbers, a, e, y, u, whereof $a=\frac{y+3u-ue}{7}$ will ſatisfy the Queſtion, as in the Table following.

a	e	y	u
1	1	2	2
1	6	4	3
2	5	4	5
&c.	*ad infinitum.*		

PRO-

Problem X.

665. To find what particular Quantities of two or more Ingredients (whose Prices are given) must be mixed with a given Quantity of another Ingredient, whose Price is also given, so that the whole may be sold at a given mean Price.

Ex. gr. What particular Quantities of Tobacco at 16 *d.* 10 *d.* and 6 *d.* the Pound may be mixed with 50 lb Weight of Tobacco at 8 *d.* the Pound, so that the whole may be sold at 12 *d.* the Pound.

Put a for the Quantity at 16 *d.* the Pound, e for that at 10 *d.* the Pound, and u for that at 6 *d.* the Pound.

$$\begin{array}{l|l|l}
\text{Then} & 1 & 16a+10e+8\times50+6u=12\times\overline{a+e+50+u} \\
 & 2 & 12a+12e+2\times50+12\times u=12\times a+e+50+u \\
1-2 & 3 & 4a-2e-4\times50-6u=0 \\
3\div2 & 4 & 2a=e+2\times50+3u \\
4\div2 & 5 & a=\frac{e+3u+100}{2}=\frac{e+3u}{2}+50
\end{array}$$

Therefore any two Numbers may be assumed for e and u, whereof $e+3u$ is divisible by 2; or any three Numbers a, e, u will satisfy the Question, whereof $a=\frac{e+3u}{2}+50$ as follows

a	e	y	u
60	11	50	3
80	30	50	10
80	9	50	17
80	18	50	14

&c. ad infinitum.

Scholium V.

666. The two last Problems are the same with what in Vulgar Arithmetic is named *Alligation Alternate*, and the latter is distinguished by the Name of *Alligation Partial*, because the Quantity of one of the Ingredients is given.

Problem XI.

667. To find an unknown Number a (an Integer, if it may be) which being multiplied by a given Integer M, and that Product divided by another given Integer N, will leave for the Remainder a third given Integer R. Or in other Terms, it is required to determinate whether $Ma-R$ be divisible by N, without a Remainder.

Effection.

1. Assume the least Integer, which being multiplied by M will make it exceed N, and call it P.

2. Seek the Remainders of these Divisions, *viz.* $\frac{M\times P}{N}$, $\frac{M\times\overline{P+1}}{N}$ putting A for the Remainder of $\frac{M\times\overline{P+0}}{N}$, B for that of $\frac{M\times\overline{P+1}}{N}$, C for that of $\frac{M\times\overline{P+2}}{N}$, D for that of $\frac{M\times\overline{P+3}}{N}$, &c. Then

3. Because these Remainders, *viz.* A, B, C, D, &c. so long as each following one exceeds that immediately foregoing, are a Series of Terms in $\div$, whose first Term is A and common Difference $B-A$; therefore, if you put $B-A=X$ when B is greater than A, or $B+N-A=X$ when B is lesser than A, by the continual Addition of X to B (subtracting N out of the Sum, as often as it exceeds N) you will have the Remainders C, D, E, F, &c. till you come to a Remainder equal to R proposed, if any such be to be found.

4. Add Unity to $P+1$, as often as you add X to B, and the last Addition will make P equal to a the unknown Integer required in its least Value. Or to shorten the Work,

5. Seek if you can find an Integer equal to R by the Addition or Subtraction of any two or more of the first three or more Remainders (A, B, C, &c.) found as above; and the Sums of their respective Integers, by which M is multipled, will be a required.

Example 1. What Number is that a, which if multiplied by $21=M$, and divided by $17=N$ the Remainder will be $11=R$? Here $\frac{21\times1}{17}$ leaves $4=A$, $\frac{21\times2}{17}$ leaves $8=B=2A$; therefore $P=1$, $X=A=4$. (Pre. 1 and 2.)

Then	1	$A=A=4$	P
$1+X$	2	$B=2A=8$	$P+1$
$2+X$	3	$C=3A=12$	$P+2$
$3+X$	4	$D=4A=16$	$P+3$
$4+X-N$	5	$E=5A-N=3$	$P+4$
$2+5$	6	$B+E=7A-N=11=R$ (Pre. 5.)	$2P+5=7=a$ in its least Value.

Example 2. What Number is that a, which if multiplied by $25=M$, and divided by $7=N$, the Remainder will be $6=R$? Here $\frac{25\times1}{7}$ leaves $4=A$, $\frac{25\times2}{7}$ leaves $1=B$; therefore $P=1$, and $X=B+N-A=4$. Then

	1	$A=4$	P
$A+X-N$	2	$B=1$	$P+1$
$2+X$	3	$C=5$	$P+2$
$2+3$	4	$B+C=6=R$	$2P+3=5=a$ in its least Value.

Example 3. What Number is that a, which if multiplied by $121=M$, and divided by $49=N$, the Remainder will be $17=R$? Here $\frac{121\times1}{49}$ leaves $23=A$, $\frac{121\times2}{49}$ leaves $46=B=2A$; therefore $P=1$ and $X=A=23$. Then

	1	$A=23$	P
$1+X$	2	$B=46$	$P+1$
$2+X-N$	3	$C=20$	$P+2$
A, C, E, in $\div$ Com. Differ. $=3$	4	$E=17=R$	$P+4=5=a$ in its least Value.

Example 4. What Number is that a, which if multiplied by $23=M$, and divided by $37=N$, the Remainder will be $30=R$? Here $\frac{23\times2}{37}$ leaves $9=A$, $\frac{23\times3}{27}$ leaves $32=B$; therefore $P=2$, and $X=B-A=23$.

$A=9$

	1	$A=9$	P
$1+X$	2	$B=32$	$P+1$
$2+X-N$	3	$C=18$	$P+2$
$3+X-N$	4	$D=4$	$P+3$
$3+4$	5	$C+D=22$	$2P+5$
$5\times\overline{2}-X$	6	$2C+2D=7$	$4P+10$
$6+X$	7	$2C+2D+X=30=R$	$4P+10+1=19=a$ in its least Value.

Example 5. What Number is that a, which if multiplied by $6=M$, and divided by $21=N$, will leave $4=R$?

Here $\frac{6\times4}{21}$ leaves $3=A$, $\frac{6\times5}{21}$ leaves $6=B=2A$; therefore $R=4$, and $X=B-A=A=3$.

	1	$A=3$	P
$1+X$	2	$B=6$	$P+1$
$2+X$	3	$C=9$	$P+2$
$1\times\overline{7}-N$	4	$7A=21$, $21-21=0$	&c.

It is therefore apparent that the Remainder will always be some one of these Numbers 3, 6, 9, 12, 15, 18, and consequently can never be 4, as is required; therefore is no such Integer as the Question supposes.

PROBLEM XII.

668. What two Integers are those a, e, whereof b times the former less c times the latter is equal to a given Integer d: Or $ba-ce=d$?

Example 1. Suppose $b=23$, $C=37$, and $d=30$.

Then	1	$23a-37e=30$ by the Question.
Whence	2	$e=\frac{23a-30}{37}$

Therefore a must be some Number, which multiplied by 23, and divided by 37, leaves 30. Consequently $a=19$ in its least Value (In. 667.) according to which the least Value of e is $\frac{23a-30}{37}=11$. And by continually adding 37 to $a=19$, and 23 to $e=11$, we shall have Answers to the Question *ad infinitum*, as follow.

| a | e |

a	e
19	11
56	34
93	57
&c.	&c.

Example 2. Suppose $b=39$, $c=56$, and $d=20$.

Then | 1 | $39a-56e=20$ by the Question.

Whence | 2 | $e=\frac{39a-20}{56}$

Therefore a must be some Number, which multiplied by 39, and divided by 56, leaves 20. Consequently the least Value of a is 12, (In. 667.) according to which the least Value of e is $\frac{39a-20}{56}=8$. And by continually adding 56 to $a=12$, and 39 to $e=8$, we shall have answers to the Question *ad infinitum.*

a	e
12	8
68	47
124	86
&c.	&c.

Scholium VI.

669. But this last Problem will be more readily answered by seeking the lesser unknown Number first, especially when the Difference between the unknown Numbers is great; in which case proceed as follows. *Ex. gr.* If $23a-37e=30$, then by due Reduction $a=\frac{37e+30}{23}$, which shews that e must be some Number, which multiplied into 37, and divided by 23, will leave $2\times23-30=16$: If $23a-37e=60$, or $a=\frac{37e+60}{23}$, then e must be some

Number,

Number, which multiplied into 37, and divided by 23, will leave $3\times23-60=9$: If $23a-37e=90$, or $a=\frac{37e+90}{23}$, then e must be some Number, which multiplied into 37, and divided by 23, will leave $4\times23-90=2$: &c. And in general, whenever the Divisor N is less than the absolute Number in the Dividend, it must be multiplied into the least Integer that makes it greater, and the said absolute Number subtracted from the Product will give the Remainder R, to be proceeded with as in In. 667. But if N be greater, the Difference between it and the absolute Number will be the Remainder R. *Ex. gr.* If $39a-56e=5$, or $a=\frac{56e+5}{39}$, then e is some Number which multiplied into 56, and divided by 39 will leave $39-5=34$. If $39a-56e=10$, or $a=\frac{56e+10}{39}$, then e is some Number, which multiplied into 56, and divided by 39, will leave $39-10=29$. If $39a-56e=15$, or $a=\frac{56e+15}{39}$ then e is some Number, which multipled into 56, and divided by 39, will leave $39-15=24$. If $39a-56e=20$, or $a=\frac{56e+20}{39}$, then e is some Number, which multiplied into 56, and divided by 39, will leave $39-20=19$, &c.

PROBLEM XIII.

670. To find whether a given Integer be composed of two or more given Integers, and if it be, to shew how many ways.

Example 1. To find what two Integers those are a, e, whereof 21 times the former, added to 17 times the latter, equals 2000. Or in other Terms, It is required to find how many Ways 100 *l.* or 2000 *s.* may be paid by Guineas of 21 *s.* the Piece, and Pistols of 17 *s.* the Piece.

	1	$21a+17e=2000$ by the Question.
$1-21a$	2	$17e=2000-21a$
$2\div17$	3	$e=117+\frac{11-21a}{17}$

Hence a must be some Number, which multiplied into 21, and divided by 17, will leave 11; therefore the least Value of a is 7, (In. 667.) and consequently the greatest Value of e is $117+\frac{11-21\times7}{17}=109$.

Then by continually adding 17 to $a=7$, and substracting 21 from $e=109$, there will be found five other Answers to this Question in Integers, as in the Table following.

| a | e |

a	e
7	109
24	88
41	67
58	46
75	25
92	4

Otherwise we may begin with finding the least Value of e, and greatest Value of a, thus;

	1	$21a+17e=2000$ by the Question.
$1-17$	2	$21a=2000-17e$
$2\div21$	3	$a=95+\frac{5-17e}{21}$

Which shews e to be some Number, which being multiplied by 17, and divided by 21, will leave 5, therefore the least Value of e is 4 (In. 667.) and consequently the greatest Value of a is $95+\frac{5-17\times4}{21}=92$, as in the Table above.

Example 2. It is required to pay 61 *l.* 7 *s.* or 1227 *s.* in Moidores of 1 *l.* 6 *s.* 6 *d.* the Piece, Pistols of 17 *s.* 6 *d.* the Piece, and Pieces of eight of 4 *s.* 6 *d.* the Piece. Or in other Terms, It is required to find three Integers a, e, y, whereof $26.5a+17.5e+4.5y$ may equal 1227.

i. e.	1	$26.5a+17.5e+4.5y=1227$
1×2	2	$53a+35e+9y=2454$
Subst.	3	$y=1$ (In. 663.)
Then	4	$53a+35e+9=2454$
Whence	5	$e=69+\frac{30-53a}{35}$

According to which, the least Value of a is 60 by the last Example, and the greatest Value of e is $69+\frac{30-53\times60}{35}=-21$ a defective Number; consequently

fequently y in the third Step was affumed too little. Therefore for a fecond Trial

Subft.	3	$y=2$
Then	4	$53a+35e+9\times2=2454$
Whence	5	$e=69+\frac{21-53a}{35}$

According to which the leaft Value of a is 42, and the greateft Value of e is $69+\frac{21-53\times42}{35}=6$.

Again, make $y=2\times2=4$, and the leaft Value of a will be 41, and $e=7$.

Therefore if 1 be continually fubtracted from $a=42$, and added to $e=6$, and 2 as often added to $y=2$, we fhall have forty two different Anfwers to the foregoing Queftion, as follows.

a	e	y
42	6	2
41	7	4
40	8	6
39	9	8
38	10	10
&c.	&c.	&c.
1	47	48

And if we fubftitute for e and for a, as has been done for y, we fhall have yet many different Anfwers peculiar to each Subftitution, which the Learner may purfue at his Leifure.

PROBLEM. XIV.

671. To find two Numbers, which added each to all their own Aliquot Parts do make the fame Sum.

Effection.

For one Number put ma, and for the other ne, and let $1+m$ and $1+n$ denote the Sums of all the Aliquot Parts of m and n refpectively.

Then

$$\left.\begin{array}{r|l|l} \text{Then} & 1 & 1+m+a+ma=1+n+e+ne \\ \text{Or} & 2 & 1+m+\overline{1+m}\times a=1+n+\overline{1+n}\times e \end{array}\right\}\text{ by the Question.}$$

$$\begin{array}{r|l|l} \text{Whence} & 3 & a=\dfrac{1+n+\overline{1+n}\times e-1-m}{1+m} \end{array}$$ *Wolfius Elem. Anal.* p. 375.

If then for m and n be assumed any two Powers of 2, $n>m$, and if for e be assumed such a Number that $e+1$ is divisible by all the Aliquot Parts of m, or by $1+m$, you will have the Value of a, and consequently the Numbers sought, *viz.* ma and ne. *Q. E. E.*

Ex. gr. If $m=2$, $n=4$, $e=5$, then $1+m$ or the Sum of the Aliquot Parts of m is $1+2$, and $1+n$, or the Sum of the Aliquot Parts of n is $1+2+4$, whence $a=\dfrac{1+2+4+\overline{1+2+4}\times 5-1-2}{1+2}=13$: Consequently $ma=26$, and $ne=20$.

Proof. $1+2+13+26=1+2+4+5+10+20$.

Again, If $m=4$, $n=8$, $e=13$; then $a=\dfrac{1+2+4+8+\overline{1+2+4+8}\times 13-1-2-4}{1+2+4}$ $=29$; consequently $ma=116$, and $ne=104$.

Proof. $1+2+4+29+58+116=1+2+4+8+13+26+52+104$.

Problem XV.

672. To find a *Perfect Number*, or one equal to the Sum of all its Aliquot Parts (In. 37.)

Effection.

Let $a^n e$ represent the Number sought, e being an incomposite Number, whence the Sum of all its Aliquot Parts is expressed by $1+a+a^2+a^3+a^4$ &c. $+e+ae^2+ae^3+ae^4$ &c. 'till the Exponent be $n-1$. But the Sum of a Series of Terms in $\div\!\div$, whose least Term is 1, Nominator of the Ratio a, and greatest Term a^{n-1} is $\dfrac{a^n-1}{a-1}$; and the Sum of a Series of Terms in $\div\!\div$, whose least Term is e, Nominator of the Ratio a, and greatest Term $a^{n-1}e$ is $\dfrac{ea^n-1}{a-1}$ (In. 603.) *i. e.* $\dfrac{a^n-1+ea^n-e}{a-1}=1+a+a^2+a^3+a^4$ &c. $+e+ae^2+ae^3+ae^4$ &c. till the Exponent be $n-1$.

Or	1	$\frac{ea^n - e + a^n - 1}{a - 1} = a^n e$ by the Question.
$1 \times \overline{a - 1}$	2	$ea^n - e + a^n - 1 = a^{n+1}e - a^n e$
$2 - \overline{a^n e - e}$	3	$a^n - 1 = a^{n+1}e - 2a^n e + e$
$3 \div \overline{a^{n+1} - 2a^n + 1}$	4	$\frac{a^n - 1}{a^{n+1} - 2a^n + 1} = e$

Now that e may be always an Integer, it is plain the last Divisor $a^{n+1} - 2a^n + 1$ must always equal Unity.

Therefore make	5	$a^{n+1} - 2a^n + 1 = 1$
$5 - 1$	6	$a^{n+1} = 2a^n$
$6 \div a^n$	7	$a = 2$
Substit. $2 = a$	8	$1 + 2 + 4$ &c. $+ e + 2e + 4e$ &c. $= 2^n e$ by the Question.
Whence	9	$\frac{1 + 2 + 4 \text{ \&c.}}{2^n - 1 - 2 - 4 \text{ \&c.} - 2^{n-1}} = e.$
But	10	$2^n - 1 - 2^1 - 2^2 - 2^3$ &c. $- 2^{n-1} = 1$
$\therefore$	11	$1 + 2 + 4 + 8$ &c. $= e$ Incomposit.

Hence we have the following Theorem for finding a perfect Number.

Theorem.

If a Series of Numbers proceeding from Unity in a twofold Ratio (*viz.* 1, 2, 4, 8, 16, &c.) be continued till their Sum be an imposit Number, that Sum multiplied into the greatest Term will make a Perfect Number.

E. gr. $1 + 2 = 3$, $2 \times 3 = 6$. $1 + 2 + 4 = 7$, $4 \times 7 = 28$. $1 + 2 + 4 + 8 + 16 = 31$, $16 \times 31 = 496$. | $31 + 32 + 64 = 127$, $64 \times 127 = 8128$. $127 + 128 + 256 = 511$, $256 \times 511 = 130816$. $511 + 512 + 1024 = 2057$, $1024 \times 2057 = 2096128$. &c.

CHAP. II.

Of Double, Triple, &c. *Lateral Equalities.*

Problem XVI.

673. To find three Numbers a, e, y, whereof the Difference between the first and second is equal to the Sum of the second and third, and the Sum of the first and third is equal to the Product of the second and third.

 Effection.

Effection.

	1	$a-e=e+y$	by the Queſtion.
	2	$a+y=ey$	
$1+e$	3	$a=2e+y$	
$2-y$	4	$a=ey-y$	
3, 4	5	$ey-y=2e+y$	
Whence	6	$y=\frac{2e}{e-2}=2+\frac{4}{e-2}$.	

Therefore to anſwer the Queſtion in Integers, for e can be aſſumed no other Numbers than 3 or 4. *Ex. gr.* If $e=3$, then $y=2+\frac{4}{3-2}=6$, (Step. 6.) and $a=2e+y=12$. (Step. 3.) If $e=4$, then $y=2+\frac{4}{4-2}=4$, and $a=2e+y=12$.

PROBLEM XVII.

674. To find three Integers a, e, y, whoſe Sum is equal to the Product of the ſecond and third, and the Sum of the firſt and third is equal to the ſecond multiplied into a given Integer b.

Effection.

	1	$a+e+y=ey$	by the Queſtion.
	2	$a+y=be$	
$1-e-y$	3	$a=ey-e-y$	
$2-y$	4	$a=be-y$	
3, 4	5	$ey-e=be$	
$5\div e$	6	$y-1=b$	
$6+1$	7	$y=b+1$ Determinate	
2, 7,	8	$a+b+1=be$	
$8-a$	9	$b+1=be-a$	

Therefore for e and a may be aſſumed any two Integers, whereof $be-a=b+1=y$. *Ex. gr.* If $b=6$, $y=b+1=7$, (Step. 7.) then e may $=2$, and conſequently $a=5$ (Step. 4.) or e may $=3$, and conſequently $a=11$; or e may $=4$, and conſequently $a=17$. &c.

PROBLEM XVIII.

675. To find three Integers a, e, y, whereof the firſt leſs the third is equal to the ſecond multiplied into a given Integer b, and the firſt divided by the third is equal to the ſecond.

Effection.

Effection.

	1	$a-y=be$ } by the Question.
	2	$\frac{a}{y}=e$
$1+y$	3	$a=be+y$
$2\times y$	4	$a=ye$
3, 4	5	$ye=be+y$
$5-y$	6	$ye+y=be$
$6\div\overline{e-1}$	7	$y=b+\frac{b}{e-1}$

Therefore for e may be assumed any Integer, which having Unity subtracted from it, will divide or measure b. *Ex. gr.* If $b=18$, then may $e=3$, according to which $y=27$ (Step 7.) and $a=81$ (Step. 4.): Or e may $=4$, according to which $y=24$, and $a=96$: Or e may $=7$, $y=21$, $a=147$: Or $e=10$, $y=17$, $a=170$.

Problem XIX.

676. **To find what particular Quantities of two or more Ingredients (whose Sum and particular Prices are given) will compose a Mixture that may be sold at a given mean Price.**

Example 1. Suppose three Sorts of Liquor were to mixed together of 3 *s.* (or 36 *d.*) 2 *s.* (or 24 *d.*) and 8 *d.* the Quart: It is required to find how much of each Sort must be taken to compose a Mixture of 129 Gallons, or 516 Quarts, to be sold at 2 *s.* (or 24 *d.*) the Quart. Or in the other Terms: It is required to divide 24 into three such Integers a, e, y, so that $36a+24e+8y$ equals 516.

	1	$a+e+y=24$ } by the Question.
	2	$36a+24e+8y=516$
1×36	3	$36a+36e+36y=864$
$3-2$	4	$12e+28y=348$
$4-28y$	5	$12e=348-28y$
$5\div 12$	6	$e=29-\frac{7}{3}y$
1, 6,	7	$a+29-\frac{7}{3}y+y=24$
Whence	8	$a=\frac{4}{3}y-7$
$\therefore$	9	$\frac{4}{3}y>5$ (In. 22.)

9×3

$9\times\overline{3}$	10	$4y>15$
$10\div4$	11	$y>3\frac{3}{4}$
6,	12	$\frac{7}{3}y<29$ (In. 22.)
$12\div\frac{7}{3}$	13	$y<12\frac{3}{4}$

It appears then (from Step. 11, and 13.) that y muſt be ſome Integer between $3\frac{3}{4}$ and $12\frac{3}{4}$; and ſuch an one as is diviſible by 3: (Step. 6, and 8.) But there are only three ſuch Integers, *viz.* 6, 9, 12; therefore the Queſtion can only admit of three Anſwers in Integers, as in the following Table.

a	*e*	*y*
3	15	6
7	8	9
11	1	12

Example 2. Suppoſe three ſorts of Grain were to be mixed together of 4*d.* $\frac{1}{2}$*d.* and $\frac{1}{4}$*d.* the Pound; It is required to find how much of each Sort muſt be taken to compoſe a Mixture of 20 Pound Weight, to be ſold at a Penny the Pound. Or in other Terms: It is required to buy 20 Fowls for 20 Pence, Geeſe, Quails and Larks, Geeſe at Groats, Quails at Half Pence, and Larks at Farthings: How many muſt there be of each Sort?

For the Number of Geeſe put a, of Quails e, and of Larks y.

Then	1	$a+e+y=20$ } by the Queſtion.
	2	$4a+\frac{1}{2}e++\frac{1}{4}y=20$ }
$2\times\overline{4}$	3	$16a+2e+y=80$
$3-1$	4	$15a+e=60$
$4-e$	5	$15a=6-e$
$5\div15$	6	$a=4-\frac{1}{15}e$
1, 6,	7	$4-\frac{1}{15}e+e+y=20$
Whence	8	$y=16-\frac{14}{15}e$
∴	9	$\frac{14}{15}e<16$ (In. 22.)
$9\div\frac{14}{13}$	10	$e<17\frac{1}{7}$

That is, e is ſome Number leſſer than $17\frac{1}{7}$, diviſible by 15 (Step. 6, 8.) but there is no ſuch Integer but 15, therefore the Queſtion can only have one Anſwer in Integers, *viz.* $e=15$, $a=3$, $y=2$.

3 *Example*

Example 3. Suppose four sorts of Wine were to be mixed together, of 16 *d.* 10 *d.* 8 *d.* and 6 *d.* the Quart: It is required to find how much of each sort must be taken to compose a Mixture of 100 Quarts, to be sold at 12 *d.* the Quart.

For the Quantity of Wine at 16 *d.* the Quart put a, at 10 *d.* e, at 8 *d.* y, at 6 *d.* u.

Then	1	$a+e+y+u=100$ } by the Question.
	2	$16a+10e+8y+6u=12\times100$ } by the Question.
$1-a$	3	$e+y+u=100-a$
$2-16a$	4	$10e+8y+6u=1200-16a$
3×6	5	$6e+6y+6u=600-6a$
3×10	6	$10e+10y+10u=1000-10a$
$4-5$	7	$4e-2y=600-10a$
Whence	8	$10a<600$
$8\div10$	9	$a<60$
$6-4$	10	$2y+4u=6a-200$
Whence	11	$6a>200$
$11\div6$	12	$a>33\frac{1}{3}$

It appears then from the 9th and 12th Steps, that a may be any Number between $33\frac{1}{3}$ and 60. Make $a=47$.

Then	13	$47+e+y+u=100$ } by the Question.
	14	$752+10e+8y+6u=1200$ } by the Question.
$13-47$	15	$e+y+u=53$
$14-752$	16	$10e+8y+6u=448$
15×10	17	$10e+10y+10u=530$
$17-16$	18	$2y+4u=82$
$18\div2$	19	$y+2u=41$
$19-2u$	20	$y=41-2u$
15, 20	21	$e+41-2u+u=53$
Whence	22	$e=12+u$
20,	23	$2u<41$
$23\div2$	24	$u<20\frac{1}{2}$

From the 22d Step it appears that there is no Limit to shew above what u ought to be taken, but in the 24th Step it is limited to some Number below $20\frac{1}{2}$; therefore, by making $a=47$, the Question will admit of 20 Answers in Integers, according to the following Table.

e | y

e	y	u
32	1	20
31	3	19
30	5	18
29	7	17
28	9	16
27	11	15
26	13	14

e	y	u
25	15	13
24	17	12
23	19	11
22	21	10
21	23	9
20	25	8
19	27	7

e	y	u
18	29	6
17	31	5
16	33	4
15	35	3
14	37	2
13	39	1

And if for a be substituted all the Integers between $33\frac{1}{3}$ and 60, then the Sum of all the Answers in Integers, which this Question admits of, will be found to be 314.

Scholium VII.

677. This last Species of Double Lateral Equalities is the same with that which in Vulgar Arithmetic is termed *Alternate Alligation Total.*

Problem XX.

678. To find four Integers a, e, y, u, whereof the Sum of the first and second is equal to the Sum of the third and fourth; the Sum of the first and third is equal to twice the Sum of the second and fourth, and the Sum of the first and fourth is equal to thrice the Sum of the second and third.

Effection.

$$\begin{array}{r|r|l}
 & 1 & a+e=y+u \\
 & 2 & a+y=2e+2u \\
 & 3 & a+u=3e+3y \\
\hline
1-e & 4 & a=y+u-e \\
2-7 & 5 & a=2e+uy-y \\
3-u & 6 & a=3e+3y-u \\
4,\ 5, & 7 & e=\dfrac{2y-u}{3} \\
5,\ 6, & 8 & e=3u-4y \\
 & 9 & y=\frac{5}{7}u,\ \text{whence } e=\frac{u}{7}\ \text{(Step. 8.)}:\ a=\frac{11}{7}u\ \text{(Step. 4.)}
\end{array}$$

Lines 1–3: by the Question.

Therefore,

Therefore, to give an Anſwer in Integers, for u muſt be aſſumed any Multiple of 7. *Ex. gr.* If $u=7$, then $y=\frac{5}{7}u=5$, $e=\frac{1}{7}u=1$, $a=\frac{11}{7}u=11$: If $u=14$, then $y=10$, $e=2$, $a=22$: If $u=21$, $y=15$, $e=3$, $a=33$: &c.

Problem XXI.

679. To find four Integers a, e, y, u, whereof the Sum of the firſt, ſecond and third is equal to the Product of the ſecond and fourth; the Sum of the firſt, ſecond and fourth is equal to the Product of the ſecond and third; and the Sum of the ſecond, third and fourth is equal to the firſt.

Effection.

	1	$\left.\begin{array}{l} a+e+y=ue \\ a+e+u=ye \\ e+y+u=a \end{array}\right\}$ by the Queſtion.
	2	
	3	
1, 2, 3,	4	$a=ue-e-y=ye-e-u=e+y+u$
Whence	5	$e=\frac{2u+y}{y-2}=\frac{2y+u}{u-2}$
Conſequently	6	$u=y$
∴	7	$e=\frac{2u+y}{y-2}=\frac{3y}{y-2}=3+\frac{6}{y-1}$
And	8	$a=e+y+u=2y+3+\frac{6}{y-2}$

For $y=u$ muſt be aſſumed ſome Number, which wanting 2 may divide 6, but there can be no ſuch Integers but three, *viz.* 3, 4, 5, conſequently the Queſtion will admit but of three Anſwers in Integers. *Ex. gr.* If $u=y=3$, then $e=9$, $a=15$. If $u=y=4$, $e=6$, $a=14$. If $u=y=5$, $e=5$, $a=15$.

CHAP. III.

Of ſingle Quadratic, Cubic, Biquadratic, &c. *Equalities.*

Partition I.

680. THE Problems belonging to this Chapter are to be diſtinguiſhed according as their *Effection* is *Synthetical* or *Analytical*.

DEFI-

Definition I.

681. By *Synthetical Effection*, I here mean that which proceeds altogether upon Theorems already investigated, assuming all the Squares, Cubes, Biquadrates, &c. required by the Question, as known, thereby to find the Value of some one or more Numbers of one Dimension in rational Terms. Of which kind are the *Effections* of the as following Problems.

Problem XXII.

682. To find two Numbers, a the greater, and e the lesser, or $a > e$, whose Sum is equal to the Square, Cube, Biquadrate, &c. of the lesser: *i. e.* $a+e=ee$, or $a+e=eee$, or $a+e=eeee$, &c.

Effection.

For e assume $z=$ any Number taken at Pleasure; so that $ee=zz$, or $eee=zzz$, or $eeee=zzzz$, &c. Or universally $e^n=z^n$, &c.

Then	1	$a+e=e^n$ } by the Question.
Or	2	$a+z=z^n$ } by the Question.
2—z	3	$a=z^n-z$ the greater Number required.

Example 1. Let $n=2$, or $a+z=z^2$, whence $a=z^2-z$: Then, If e or $z=3$, $a=6$: Proof $a+e=ee=9$. If $z=4$, $a=12$: Proof $a+e=ee=16$. If $z=5$, $a=20$: Proof $a+e=ee=25$. &c.

Example 2. Let $n=3$, or $a+z=z^3$, whence $a=z^3-z$: Then, If e or $z=3$, $a=24$: Proof $a+e=eee=27$. If $e=4$, $a=60$. &c.

Example 3. Let $n=4$, or $a+z=z^4$, whence $a=z^4-z$: Then, if e or $z=2$, $a=14$: Proof $a+e=eeee=16$. If $e=3$, $a=78$. &c.

Problem XXIII.

683. To find two Numbers $a > e$, whose Difference is equal to the n Power of the lesser, *i. e.* $a-e=e^n$.

Effection.

Assume	1	$e=z$ any Number at pleasure.
$1 \textcircled{n}$	2	$e^n=z^n$
Then	3	$a-e=e^n$ } by the Question.
Then	4	$a-z=z^n$ } by the Question.
4+z	3	$a=z^n+z$

Example 1. Let $n=2$, or $a=z^2+z$: Then, if e or $z=2$, $a=6$. If $e=3$, a will $=12$. If $e=4$, $a=20$. &c.

Example 2. Let $n=3$, or $a=z^3+z$: Then, If e or $z=2$, $a=10$. If $e=3$, $a=30$. If $e=4$, $a=68$, &c.

Problem XXIV.

684. To divide a given Number s into two Numbers $a>e$, whose Difference d is a Number of the Power of n, or an n Power, *i. e.* is a Square, Cube, Biquadrate, &c. as you please.

Effection.

$$\begin{array}{l|l|l}
\text{Assume} & 1 & a+e=s \text{ by the Question.} \\
 & 2 & d^n=a-e<s \\
\text{Then} & 3 & a=\dfrac{s+d^n}{2} \\
 & 4 & e=\dfrac{s-d^n}{2}
\end{array}\Bigg\}\ \text{In. } 545.$$

Example 1. Let $n=2$, or $a=\frac{s+d^2}{2}$ and $e=\frac{s-d^2}{2}$: If then $s=35$, and $dd=9$, $a=22$ and $e=13$. Proof. $a-e=dd=9$.

Example 2. Let $n=3$, or $a=\frac{s+d^3}{2}$ and $e=\frac{s-d^3}{2}$: If then $s=35$, and $ddd=27$, $a=31$, and $e=4$. Proof. $a-e=ddd=27$, $a+e=s=35$.

Problem XXV.

686. To divide a given Number s into two Numbers $a>e$, whose Quotient $\frac{a}{e}$ is an n Power.

Effection.

$$\begin{array}{l|l|l}
 & 1 & a+e=s \text{ by the Question.} \\
\text{Assume} & 2 & q^n=\dfrac{a}{e} \text{ at pleasure.} \\
\text{Then} & 3 & a=\dfrac{sq^n}{q^n+1},\ e=\dfrac{s}{q^n+1}\quad (\text{In. } 548.)
\end{array}$$

Example 1. Let $n=2$. If then $s=35$ and $qq=4$, $a=28$, $e=7$. Proof. $\frac{a}{e}=qq=4$, and $a+e=s=35$.

Example 2. Let $n=3$. If then $s=35$, and $qqq=8$, $a=\frac{280}{9}$, $e=\frac{35}{9}$. Proof. $\frac{a}{e}=qqq=8$, and $a+e=s=35$.

Problem XXVI.

686. To divide a given Number s into two Numbers $a > e$, the Difference of whose Squares is an n Power.

Effection.

$$\begin{array}{r|l|l} & 1 & a+e=s \text{ by the Question,} \\ \text{Assume} & 2 & x^n=aa-ee < s \\ \text{Then} & 3 & a=\dfrac{ss+x^n}{2s},\ e=\dfrac{ss-x^n}{2s} \quad \text{In. 550.)} \end{array}$$

Example 1. Let $n=2$. If then $s=6$ and $x^2=4$, $a=\frac{10}{3}$ and $e=\frac{8}{3}$. Proof. $aa-ee=x^2=\frac{36}{9}$ or 4, and $a+e=s=6$.

Example 2. Let $n=3$. If then $s=6$ and $x^3=8$, $a=\frac{11}{3}$ and $e=\frac{7}{3}$. Proof. $a^2-e^2=x^3=\frac{72}{9}$ or 8, and $a+e=s=6$.

Problem XXVII.

687. To find two Numbers $a > e$, from their Difference $=d$ given, whose Sum shall be an n Power.

Effection.

$$\begin{array}{r|l|l} & 1 & a-e=d \text{ by the Question.} \\ \text{Assume} & 2 & s^n=a+e > d \\ \text{Then} & 3 & a=\dfrac{s^n+d}{2},\ e=\dfrac{s^n-d}{2} \quad \text{(In. 545.)} \end{array}$$

Example 1. Let $n=2$. If then $d=3$ and $s^2=25$, $a=14$, $e=11$. Proof. $a+e=25$, and $a-e=d=3$.

Example 2. Let $n=3$. If then $d=3$ *and* $s^3=125$, $a=64$, $e=61$. $a+e=s^3=125$, and $a-e=d=3$.

Problem. XXVIII.

688. To find two Numbers $a > e$, from their Quotient $=q$ given, or from the Proportion of a to e given, which let be as b to c, and the Sum of the said Numbers shall be an n Power.

Effection.

$$\begin{array}{r|l|l} & 1 & \dfrac{a}{e}=q \text{ or } \dfrac{b}{c} \text{ by the Question.} \\ \text{Assume} & 2 & s^n=a+e \text{ at pleasure.} \\ \text{Then} & 3 & a=\dfrac{qs^n}{q+1} \text{ or } \dfrac{bs^n}{b+c}\ e=\dfrac{s^n}{q+1} \text{ or } \dfrac{cs^n}{b+c} \quad \text{(In. 548.)} \end{array}$$

Example

Example 1. Let $n=2$. If $q=3$ and $s^2=4$, then $a=3$, $e=1$. Proof. $a+e=4$, and $\frac{a}{e}=3$.

Example 2. Let $n=3$. If $q=3$ *and* $s^3=8$, then $a=6$, $e=2$. Proof. $a+e=8$, and $\frac{a}{e}=3$.

Problem XXIX.

689. To find two Numbers $a>e$, from the Difference of their Squares $=x$ given, whose Sum shall be an n Power.

Effection.

$$\text{Assume}\left|\begin{array}{l|l} 1 & aa-ee=x \text{ by the Question.} \\ 2 & s^n=a+e>x^{\frac{1}{2}} \\ 3 & a=\dfrac{s^{2n}+x}{2s^n}\quad e=\dfrac{s^{2n}-x}{2s^n} \quad \text{(In. 550.)} \end{array}\right.$$

Example 1. Let $n=2$. If then $x=45$ and $s^2=9$, $a=7$, $e=2$. Proof. $a+e=9$ *and* $a^2-e^2=45$.

Example 2. Let $n=3$. If then $x=45$ *and* $s^3=27$, $a=\frac{774}{54}$, $e=\frac{684}{54}$. Proof. $a+e=\frac{1458}{54}=27$, $a^2-e^2=\frac{131220}{2916}=45$.

Problem XXX.

690. To find two Numbers $a>e$ from their Difference $=d$ given, whose Quotient $\frac{a}{e}$ shall be an n Power.

Effection.

$$\begin{array}{r|l|l} & 1 & a-e=d \text{ by the Question.} \\ \text{Assume} & 2 & q^n=\dfrac{a}{e}>1 \\ \text{Then} & 3 & a=\dfrac{dq^n}{q^n-1},\ e=\dfrac{d}{q^n-1} \quad \text{(In. 554.)} \end{array}$$

Example 1. Let $n=2$. If then $d=15$ and $q^2=4$, $a=20$, $e=5$. Proof. $\frac{a}{e}=4$, $a-e=15$.

Example 2. Let $n=3$. If then $d=15$ and $q^3=8$, $a=\frac{120}{7}$, $e=\frac{15}{7}$. Proof. $\frac{a}{e}=8$, $a-e=\frac{105}{7}=15$.

PROBLEM XXXI.

691. To find two Numbers $a > e$ from their Quotient $\frac{a}{e}=q$ given, or which are to one another as b to c, whose Difference shall be an n Power.

Effection.

	1	$\frac{a}{e}=q=\frac{b}{c}$ by the Question.
Assume	2	$d^n=a-e$ at pleasure
Then	3	$a=\frac{qd^n}{q-1}=\frac{bd^n}{b-c}, e=\frac{d^n}{q-1}=\frac{cd^n}{b-c}$ (In. 554.)

Example 1. Let $n=2$. If then $9=3$ and $d^2=4$, $a=6$, $e=2$. Proof. $a-e=4$, $\frac{a}{e}=3$.

Example 2. Let $n=3$. If then $q=3$ and $d^3=8$, $a=12$, $e=4$. Proof. $a-e=8$, $\frac{a}{e}=3$.

PROBLEM XXXII.

692. To find two Numbers $a > e$ from their Difference $=d$ given, the Difference of whose Squares shall be an n Power.

Effection.

	1	$a-e=d$ by the Question.
Assume	2	$x^n=a^2-e^2 > d$
Then	3	$a=\frac{x^n+d^2}{2d}, e=\frac{x^n-d^2}{2d}$ (In. 556.)

Example 1. Let $n=2$. If then $d=2$ and $x^2=9$, $a=\frac{13}{4}$, $e=\frac{5}{4}$. Proof. $a^2-e^2=\frac{144}{16}=9$, $a-e=\frac{8}{4}=2$.

Example 2. Let $n=3$. If then $d=2$ *and* $x^3=27$, $a=\frac{31}{4}$, $e=\frac{23}{4}$. Proof. $a^2-e^2=\frac{432}{16}=27$, $a-e=\frac{8}{4}=2$.

PROBLEM XXXIII.

693. To find two Numbers $a > e$ from the Difference of their Squares $=x$ given, whose Difference shall be an n Power.

Effection

Effection.

$$\begin{array}{r|l|l} & 1 & aa-ee=x \text{ by the Question.} \\ \text{Assume} & 2 & d^n=a-e<x^{\frac{1}{2}} \\ \text{Then} & 3 & a=\dfrac{x+d^{2n}}{2d^n},\ e=\dfrac{x-d^{2n}}{2d^n} \quad \text{(In. 556.)} \end{array}$$

Example 1. Let $n=2$. If then $x=99$ and $d^2=9$, $a=10$, $e=1$. Proof. $a-e=9$, $aa-ee=1$.

Example 2. Let $n=3$. If then $x=99$ and $d^3=8$, $a=\frac{163}{16}$, $e=\frac{35}{16}$. Proof. $a-e=\frac{128}{16}=8$, $a^2-e^2=\frac{25344}{256}=99$.

Problem XXXIV.

694. To find two Numbers $a>e$ whose Sum is an n Power, and Difference an m Power.

Effection.

$$\text{Assume}\begin{cases} s^n=a+e \text{ at pleasure.} \\ d^m=a-e<s^n \end{cases}$$

$$\text{Then } a=\frac{s^n+d^m}{2},\ e=\frac{s^n-d^m}{2} \quad \text{(In. 545.)}$$

Example 1. Let $n=m=2$. If then $s^2=25$ and $d^2=9$, $a=17$, $e=8$. Proof. $a+e=5^2$, $a-e=3^2$.

Example 2. Let $n=2$, $m=3$. If then $s^2=25$ and $d^3=8$, $a=\frac{33}{2}$, $e=\frac{17}{2}$. Proof. $a+e=5^2$, $a-e=2^3$.

Problem XXXV.

695. To find two Numbers $a>e$, whose Sum is an n Power, and Quotient an m Power.

Effection.

$$\text{Assume}\left\{\begin{aligned} s^n&=a+e \\ q^m&=\frac{a}{e} \end{aligned}\right\} \text{at pleasure.}$$

$$\text{Then } a=\frac{q^m s^n}{q^m+1},\ e=\frac{s^n}{q^m+1} \quad \text{(In. 548.)}$$

Example 1. Let $n=m=3$. If then $s^3=8$, and $q^3=27$, $a=\frac{216}{28}=\frac{54}{7}$, $e=\frac{8}{28}=\frac{2}{7}$.

Example 2. Let $n=3$, $m=2$. If then $s^3=8$ and $q^2=9$, $a=\frac{72}{10}=\frac{36}{5}$, $e=\frac{8}{10}=\frac{4}{5}$.

PROBLEM XXXVI.

696. To find two Numbers $a > e$, whose Difference shall be an n Power, and Quotient an m Power.

Effection.

$$\text{Assume}\begin{cases} d^n = a - e \text{ at pleasure.} \\ q^m = \dfrac{a}{e} > 1 \end{cases}$$

$$\text{Then } a = \frac{d^n q^m}{q^m - 1},\ e = \frac{d^n}{q^m - 1} \quad (\text{In. } 554.)$$

Example 1. Let $n = m = 2$. If then $d^2 = 4$ and $q^2 = 9$, $a = \frac{18}{4}$, $e = \frac{2}{4}$.
Example 2. Let $n = 3$, $m = 2$. If then $d^3 = 8$ and $q^2 = 9$, $a = 9$, and $e = 1$.

PROBLEM XXXVII.

697. To find two Numbers $a > e$, whose Difference is an n Power, and Difference of their Squares an m Power.

Effection.

$$\text{Assume}\begin{cases} d^n = a - e \text{ at pleasure.} \\ x^m = a^2 - e^2 > d^{2n} \end{cases}$$

$$\text{Then } a = \frac{x^m + d^{2n}}{2d^n},\ e = \frac{x^m - d^{2n}}{2d^n} \quad (\text{In. } 556.)$$

Example 1. Let $n = m = 2$. If then $d^2 = 1$ and $x^2 = 4$, $a = \frac{5}{2}$, $e = \frac{3}{2}$.
Example 2. Let $n = 2$, $m = 3$. If then $d^2 = 4$ and $x^3 = 27$, $a = \frac{43}{8}$, $e = \frac{11}{8}$.

PROBLEM XXXVIII.

698. To find two Numbers $a > e$, whose Sum is an n Power, and Sum of their Squares an m Power.

Effection.

$$\text{Assume}\begin{cases} s^n = a + e \text{ at pleasure.} \\ x^m = aa - ee < s^{2n} \end{cases}$$

$$\text{Then } a = \frac{s^{2n} + x^m}{2s^n},\ e = \frac{s^{2n} - x^m}{2s^n} \quad (\text{In. } 550.)$$

Example 1. Let $n = m = 2$. If then $s^n = 25$ and $x^m = 9$, $a = \frac{634}{50} = \frac{317}{25}$, $e = \frac{616}{50} = \frac{308}{25}$.
Example 2. Let $n = 2$, $m = 3$. In then $s^n = 16$ and $x^m = 8$, $a = \frac{33}{4}$, $e = \frac{31}{4}$.

PRO-

PROBLEM XXXIX.

699. To find three Numbers a, e, y, whereof the Sum of the firſt and ſecond is an n Power, the Sum of the firſt and third an m Power, and the Sum of the ſecond and third an l Power.

Effection.

$$\text{Aſſume}\begin{cases} b^n = a + e < c^m + d^l \\ c^m = a + y < b^n + d^l \\ d^l = e + y < b^n + c^m \end{cases}$$

$$\text{Then } a = \frac{b^n + c^m - d^l}{2},\ e = \frac{b^n + d^l - c^m}{2},\ y = \frac{c^m + d^l - b^n}{2} \quad \text{(In. 575.)}$$

Ex. gr. Let $n = m = 2$, $l = 3$. If then $b^2 = 4$, $c^2 = 9$, and $d^3 = 8$: a will $= \frac{5}{2}$, $e = \frac{3}{2}$, $y = \frac{13}{2}$.

PROBLEM. XL.

700. To divide a given Number s into three ſuch Parts, $a > e > y$, ſo that the firſt leſs the ſecond ſhall be an n Power, and the ſecond leſs the third an m Power.

Effection.

Here, becauſe I have no Theorem to reſolve this Queſtion by, therefore I ſeek what three Numbers thoſe are $a > e > y$, whereof $a + e + y = s$, $a - e = b$, and $e - y = c$: And I find that $a = \frac{s + 2b + c}{3}$, $e = \frac{s + c - b}{3}$, and $y = \frac{s - 2c - b}{3}$. Therefore

$$\text{Aſſume}\begin{cases} b^n = a - e < s - 2c^m \\ c^m = e - y < \dfrac{s - b^n}{2} \end{cases}$$

$$\text{Whence } a = \frac{s + 2b^n + c^m}{3},\ e = \frac{s + c^m - b^n}{3} \text{ and } y = \frac{s - 2c^m - b^n}{3}. \quad Q.E.E.$$

Ex. gr. Let $n = 3$, $m = 2$. If then $s = 28$, $b^3 = 8$, and $c^2 = 9$: $a = \frac{53}{3}$, $e = \frac{29}{3}$, $y = \frac{2}{3}$.

PROBLEM XLI.

701. To divide a given Number s into four ſuch Parts $a > e > y > u$, ſo that the Difference between the firſt and ſecond ſhall be an n Power, between the ſecond and third an m Power, and between the third and fourth an l Power.

Effection

Effection.

$a+e+y+u=s$ by the Queſtion.

$$\text{Aſſume}\begin{cases} b^n=a-e<s+3d^l-2c^m \\ c^m=e-y<\dfrac{s-b^n-3d^l}{2} \\ d^l=y-u<\dfrac{s-b^n-2c^m}{3} \end{cases}$$

$$\text{Then}\begin{cases} a=\dfrac{s+3b^n+2c^m+d^l}{4},\ e=\dfrac{s+2c^m+d^l-b^n}{4},\ y=\dfrac{s+d^l-2c^m-b^n}{4}, \\ u=\dfrac{s-b^n-2c^m+3d^l}{4}. \quad \text{(In. 576.)} \end{cases}$$

Ex. gr. Let $n=l=2$, $m=3$. If $s=57$, $b^2=9$, $c^3=8$, $d^2=4$, then $a=26$, $e=17$, $y=9$, $u=5$.

Scholium VIII.

702. From theſe Examples I ſuppoſe the Learner will readily diſcover how to make the like Uſe of any other Theorems whatever, where the unknown Quantities are expreſſed in rational Terms.

Definition II.

703. By *Analytical Effection* I mean that which takes the Squares, Cubes, Biquadrates, &c. that are mentioned in the Queſtion, as unknown Terms; and as ſuch proceeds by Reduction to ſeek the Value of their Roots in known rational Terms.

Partition II.

704. *Analytical Effection* is divided into *Natural* and *Artificial*.

Definition III.

705. *Natural Analyſis* is that which, without any contrived Hypotheſis, reduces the Numbers or Quantities ſought to Laterals, or Numbers of one Dimenſion, like the 4th, 5th and 6th Problems of the firſt Chapter of this Part, of which kind again are the three following Problems.

Problem XLII.

706. To find two Numbers a, e, whereof the latter added to the Square of the former ſhall make a Square Number, whoſe Root is equal to the Sum of the Numbers ſought.

Effection.

Effection.

	1	$a^2+e=a^2+2ae+e^2$ by the Question.
$1-a^2$	2	$e=2ae+ee$
$2\div e$	3	$1=2a+e$
$3-2a$	4	$1-2a=e$
Whence	5	$\frac{1-e}{2}=a.$

Therefore the Numbers fought muft always be lefs than Unity. *Ex. gr.* If $e=\frac{1}{2}$, $a=\frac{1}{4}$. If $e=\frac{1}{3}$, $a=\frac{2}{6}$. If $e=\frac{1}{4}$, $a=\frac{3}{8}$. If $e=\frac{1}{5}$, $a=\frac{4}{10}$.

Problem XLIV.

707. To find two Numbers $a>e$, whofe Difference is to the Difference of their Squares as b to c.

Effection.

	1	$a-e:aa-ee=b:c.$ or $b\times\overline{aa-ee}=c\times\overline{a-e}.$ by the Queftion.
$1\div\overline{a-e}$	2	$ba+be=c.$ Whence $a=\frac{c}{b}-e.$

Ex. gr. If $\frac{c}{b}=9$ and $e=4$, $a=5$. If $\frac{c}{b}=9$ and $e=3$, $a=6$. If $\frac{c}{b}=9$ and $e=2$, $a=7$, &c.

Problem XLVI.

708. To find two Numbers $a>e$, the Sum of whofe Cubes is equal to the Sum of the Cube and Square of the greater multiplied into the leffer, lefs the Difference of the Cube and Square of the leffer multiplied into the greater.

Effection.

	1	$a^3+e^3=\overline{a^3+a^2}\times e-\overline{e^3-e^2}\times a$ by the Queftion.
Or	2	$a^3+e^3=a^3e+a^2e-e^3a+e^2a$
$2-a^2e-e^3$	3	$a^3-a^2e=a^3e+ae^2-ae^3-e^3$
$3\div\overline{a-e}$	4	$a^2=a^2e+ae^2+e^2$
$4-e^2$	5	$a^2-e^2=a^2e+ae^2$
$5\div\overline{a+e}$	6	$a-e=ae$
Whence	7	$a=\frac{1}{1-e}-1$

Therefore e muft be lefs than Unity. *Ex. gr.* If $e=\frac{1}{2}$, $a=1$. If $e=\frac{1}{3}$, $a=\frac{1}{2}$. If $e=\frac{1}{4}$, $a=\frac{1}{3}$. If $e=\frac{1}{5}$, $a=\frac{1}{4}$, &c.

DEFINITION IV.

709. An *Artificial Analysis* is that whereby the Numbers sought are reduced to Laterals, by some contrived Hypothesis, or Representation of the Terms of the Question.

PARTITION III.

710. And this again is twofold, *viz.* first by *Subtraction*, and secondly by *Division*.

DEFINITION V.

711. *Artificial Analysis* by *Subtraction* is that which proceeds by representing the Root of the required Power in such sort, that the highest Dimensions of the unknown Term is made to vanish from each Side of the Equation by Subtraction, as in the Effection of the five next following Problems.

PROBLEM XLV.

712. To find a Number a, which, if added to its Square, shall make a Square Number ee.

Effection.

	1	$aa+a=ee$ by the Question.
Assume	2	$r=$ any Number at pleasure.
Make	3	$r-a=e$, or $r+a=e$, or $a-r=e$
$3Q^2$	4	$rr-2ra+aa=ee$ by the first Hypothesis
1, 4.	5	$aa+a=rr-2ra+aa$
$5-aa$	6	$a=rr-2ra$
Whence	7	$a=\frac{rr}{2r+1}$, $e=\frac{rr+r}{2r+1}$

Ex. gr. If $r=1$, $a=\frac{1}{3}$. If $r=2$, $a=\frac{4}{5}$. If $r=3$, $a=\frac{9}{7}$, &c.

PROBLEM XLVI.

713. To find a Number a, which, if subtracted from its Square, will leave a Square Number ee.

	1	$aa-a=ee$ by the Question.
Assume	2	$r>1$
Make	3	$r-a=e$,
$3Q^2$	4	$rr-2ra+aa=ee$
1, 4.	5	$a^2-a=rr-2ra+aa$
$5-aa$	6	$-a=r^2-2ar$
Whence	7	$a=\frac{rr}{2r-1}$, $e=\frac{rr-r}{2r-1}$ therefore $r>1$.

Ex. gr. If $r=2$, $a=\frac{4}{3}$. If $r=3$, $a=\frac{9}{5}$, If $r=4$, $a=\frac{16}{7}$, &c.

PRO-

Problem XLVII.

714. To find a Square Number aa, from which, if a given Number d be subtracted, the Difference will be a Square Number ee. Or which is the same: To find two Square Numbers $aa > ee$, whose Difference is equal to d a Number given.

Effection.

	1	$aa - d = ee$ by the Question.
Assume	2	$rr > d$
Make	3	$r - a = e$. If $r > d$, or $r + a$ if r be lesser than d
$3 \odot^2$	4	$rr - 2ra + aa = ee$
1, 4,	5	$aa - d = rr - 2ra + aa$
$5 - aa$	6	$-d = rr - 2ra$
Whence	7	$a = \frac{rr + d}{2r}$, $e = \frac{rr - d}{2r}$, therefore $rr > d$.

Ex. gr. If $d = 24$ and $r = 30$, then $a = \frac{77}{5}$ and $e = \frac{73}{5}$.

Problem XLVIII.

715. To find a Square Number aa, to which, if a given Square Number bb be added, the Sum will be a Square Number ee.

Effection.

	1	$aa + bb = ee$ by the Question.
Assume	2	$r > b$
Make	3	$a + r = e$
$3 \odot^2$	4	$aa + 2ar + rr = ee$
1, 4,	5	$aa + bb = aa + 2ar + rr$
$5 - aa$	6	$bb = 2ar + rr$
Whence	7	$a = \frac{bb - rr}{2r}$, $e = \frac{bb + rr}{2r}$. therefore $r < b$

Ex. gr. If $bb = 9$ and $rr = 4$, $a = \frac{5}{4}$, $e = \frac{13}{4}$.

Problem. XLIX.

716. To find a Square Number a^2, from which, if a given Cube bbb be subtracted, the Remainder will be a Square ee.

Effection.

	1	$aa - bbb = ee$ by the Question.
Assume	2	$r > b^{\frac{3}{2}}$
Make	3	$r - a = e$

$3\odot^2$	4	$r^2-2ra+a^2=e^2$
1, 4,	5	$a^2-b^2=r^2-2ra+a^2$
$5-aa$	6	$-b^2=r^2-2ra$
Whence	7	$a=\frac{r^2+b^2}{2r}$, $e=\frac{r^2-b^2}{2r}$, therefore $r^2>b^2$

Ex. gr. $b^2=8$ and $r=10$, then $a=\frac{27}{5}$ and $e=\frac{23}{5}$.

Scholium IX.

717. It is apparent then, that this Analysis by Subtraction, can be only of use for discovering the Roots from Squares, since all higher Powers formed from a Binomial, are also affected with every inferior one: Neither can it be of Use for that itself, where the Term of two Dimensions is either Negative, or affected with a Coefficient which is not a Square, because in the former Case, the Term which should answer it in the assumed Square will always be positive, (In. 393.) and in the latter, it will always be a compleat Square.

Definition VI.

718. *Artificial Analysis* by Division is that which proceeds by contriving such a Representation for the Root of the required Power, as shall make all the absolute Numbers in the Equation to vanish, and consequently render it capable of Division by the unknown Term. After this manner is performed the Effections of the remaining Problems of this Chapter; except the 57th, 58th, 59th, 60th and 61st.

Problem L.

719. To divide a given Square bb into two Squares aa and ee.

Effection.

	1	$bb=aa+ee$ by the Question.
$1-aa$	2	$bb-aa=ee$
Assume	3	$r>1$
Make	4	$ra-b=e$, if $r>b$; or $b-ra=e$, if $b>r$
$4\odot^2$	5	$r^2a^2-2rab+bb=ee$
2, 5	6	$bb-aa=r^2a^2-2rab+bb$
$6-bb$	7	$-aa=r^2a^2-2rab$
$7\div a$	8	$a=rra-2rb$
Whence	9	$a=b\times\frac{2r}{rr+1}$, $e=b\times\frac{rr-1}{rr+1}$, therefore $r>1$

Ex. gr. If $b=5$ and $r=3$, then $a=3$ and $e=4$.

Corol-

Corollary I.

720. Hence it is easy to conceive how any given Square Number may be divided into any Number of Squares at pleasure. *Ex. gr.* If BB be divided into AA and EE, and EE again into YY and UU, then $BB=AA+YY+UU$ three Squares; and again, if AA be divided into aa and ee, then $BB=aa+ee+YY+UU$ four Squares, *&c.*

Problem LI.

721. To find a Square Number aa, which multiplied into a given Number b, and then added to a given Square dd, will make a Square ee.

Effection.

	1	$baa+dd=ee$ by the Question.
Assume	2	$r>b^{\frac{1}{2}}$
Make	3	$ra-d=e$
$3\odot^2$	4	$rraa-2rda+dd=ee$
1, 4,	5	$baa+dd=rraa-2rda+dd$
$5-dd$	6	$baa=rraa-2rda$
$6\div a$	7	$ba=rra-2rd$
Whence	8	$a=\frac{2rd}{rr-b}$, $e=ra-d=\frac{rrd+bd}{rr-b}$, therefore $rr>b$

Ex. gr. If $b=7$, $d=2$, and $r=3$, then $a=6$ and $e=16$.

Problem LII.

722. To divide two given Squares $bb>dd$ into two other Squares AA and EE.

Effection.

	1	$bb+dd=AA+EE$ by the Question.
Assume	2	$r>d$ and different from b
Make	3	$ba-b=A$
	4	$ra-d=E$
$3\odot^2$	5	$b^2a^2-2b^2a+b^2=A^2$
$4\odot^2$	6	$r^2a^2-2rda+d^2=E^2$
$5+6$	7	$\begin{matrix}+b^2\\+r^2\end{matrix}a^2-\begin{matrix}+2bb\\+2rd\end{matrix}a+b^2+d^2=A^2+E^2=b^2+d^2$ Step. 1.
$7-b^2-d^2$	8	$\begin{matrix}+bb\\+rr\end{matrix}a^2-\begin{matrix}+2bb\\+2rd\end{matrix}a$
$8\div a$	9	$bba+rra=2bb+2rd$
$9\div\overline{b^2+r^2}$	10	$a=\frac{2bb+2rd}{bb+rr}$
	11	$A=ba-b=\frac{b^3+2bdr-br^2}{bb+rr}$. $E=ra-d=\frac{dr^2+2b^2r-db^2}{bb+rr}$.

Ex. gr. If $b=4$, $d=2$, and $r=6$; then $a=\frac{14}{13}$, consequently $A=\frac{4}{13}$, and $E=\frac{58}{13}$.

Problem LIII.

723. To divide three given Squares $bb>cc>dd$ into three other Squares A^2, E^2, Y^2.

Effection.

	1	$bb+cc+dd=A^2+E^2+Y^2$ by the Question.
Assume	2	$r>d$
Make	3	$ba-b=A$, $ca-c=E$, and $ra-d=Y$.
$3 \circledS^2$	4	$\overline{ba-b}^2+\overline{ca-c}^2+\overline{ra-d}^2=A^2+E^2+Y^2=bb+cc+dd$ (Step. 1.
$4-b^2-c^2-d^2$	5	$\overline{bb+cc+rr}\times a^2=\overline{2bb+2cc+2rd}\times a$
Whence	6	$a=2\times\frac{bb+cc+rd}{bb+cc+rr}$

Ex. gr. If $b=4$, $c=2$, $d=1$, $r=3$, then $a=\frac{46}{29}$; and consequently $A=ba-b=\frac{68}{29}$, $E=ca-c=\frac{34}{29}$, $Y=ra-d=\frac{109}{29}$.

Problem LIV.

724. To divide a given Number s into two Parts A, E, whose Product shall be a Square Number yy.

Effection.

Put	1	$a=A-E$
Then	2	$A=\frac{s+a}{2}$ In. 545.
	3	$E=\frac{s-a}{2}$
	4	$AE=\frac{ss-aa}{4}=yy$ by the Question.
$\bar{4}\times 4$	5	$ss-aa=4yy$
Make	6	$ra-s=2y$
$6 \circledS^2$	7	$rraa-2rsa+ss=4yy=ss-aa$ Step. 5.
$7-ss$	8	$rraa-2rsa=-aa$
$8\div a$	9	$rra-2rs=-a$
Whence	10	$a=\frac{2rs}{rr+1}$. Whence $A=\frac{s+a}{2}=s\times\frac{rr+2r+1}{2rr+2}$. $E=s\times\frac{rr-2r+1}{2rr+2}$.

Ex.

Ex. gr. If $s=10$, and $r=2$, then $a=8$; consequently $A=\frac{s+a}{2}=9$, and $E=\frac{s-a}{2}=1$.

Problem LV.

725. To find two Numbers a, e, whereof the Square of the former, added to the latter, shall equal the Square Root of their Sum.

Effection.

	1	$aa+e=\overline{aa+2ae+e^2}^{\frac12}$ by the Question.
$1\odot^2$	2	$a^4+2aae+ee=a+e$.
$2-a^4$	3	$ee+\overline{2aa+1}\times e=a-a^4$
Whence	4	$e=\overline{a-a^2+\frac14}^{\frac12}-a^2+\frac12$
Make	5	$ra-\frac12=\overline{a-aa+\frac14}^{\frac12}$
$5\odot^2$	6	$rraa-ra+\frac14=a-aa+\frac14$
$6-\frac14$	7	$rraa-ra=a-aa$
$7\div a$	8	$rra-r=1-a$
Whence	9	$a=\frac{r+1}{rr+1}$. $e=ra-aa$ (Step. 4.)

Ex. gr. If $r=2$, $a=\frac35$ and $e=\frac{21}{25}$.

Problem LVI.

726. To find two Numbers $A>E$, whose Sum added to their Product is a Square Number yy.

Effection.

Put	1	$2a=$ the Sum of the two Numbers required.
Assume	2	$2d=$ their Difference
Then	3	$a+d=A$ } In. 546.
	4	$a-d=E$ }
3×4	5	$a^2-d^2=AE$
$3+4+5$	6	$a^2-d^2+2a=yy$ by the Question.
Make	7	$r-a=y$, $r<d$ (In. 711.)
$7\odot^2$	8	$rr-2ra+aa=yy$.
6, 8,	9	$a^2-d^2+2a=rr-2ra+aa$
$9-aa$	10	$-dd+2a=rr-2ra$
Whence	11	$a=\frac{dd+rr}{2r+2}$
∴	12	$A=\frac{dd+2dr+rr+2d}{2r+2}$. $E=\frac{dd-2dr+rr-2d}{2r+2}$

Or

Or otherwise $A=\frac{\overline{d+r}^2+2d}{2r+2}$, $E=\frac{\overline{d-r}^2-2d}{2r+2}$.

Ex. gr. If $d=10$. and $r=2$, then $A=$ [illegible] and $E=$ [illegible].

Problem LVII.

727. To find two Numbers $A>E$, whose Product added to the Sum of their Squares is a Square Number yy.

Effection.

Assume	1	$2b=A+E$ at pleasure.
Put	2	$2e=A-E$
Then	3	$b+e=A$ } In. 546.
	4	$b-e=E$ }
3×4	5	$bb-ee=AE$
3 ☉ ^{2}	6	$b^2+2be+e^2=A^2$
4 ☉ ^{2}	7	$b^2-2be+e^2=E^2$
$5+6+7$	8	$3b^2+ee=yy$ by the Question.
Make	9	$r+e=y$
Then	10	$3bb+ee=rr+2re+ee$
Whence	11	$e=\frac{3bb-rr}{2r}$
Therefore	12	$A=\frac{3bb+2br-rr}{2r}$, $E=\frac{rr+2br-3bb}{2r}$.

Ex. gr. If $b=4$, $r=6$, then $A=5$, and $E=3$.

Problem LVIII.

728. To divide a given Number s into two Parts, a, e, the Sum of whose Squares shall be a Square yy.

Effection.

	1	$a+e=s$ by the Question.
	2	$e=s-a$
	3	$aa=$ the Square of one Part
	4	$ss-2sa+aa=$ the Square of the other
$3+4$	5	$2aa-2sa+ss=yy$ by the Question.
Make	6	$ra-s=y$
6 ☉ ^{2}	7	$rraa-2rsa+ss=yy$
$5, 7.$	8	$2aa-2sa+ss=rraa-2rsa+ss$
$8-ss$	9	$2aa-2sa=rraa-2rsa$

$9 \div a$	10	$2a - 2s = rra - 2rs$
Whence	11	$a = \frac{2rs - 2s}{rr - 2} = s \times \frac{2r - 2}{rr - 2}$, $e = s - a = s \times \frac{rr - 2r}{rr - 2}$.

Ex. gr. If $s = 12$, and $r = 3$, then $a = \frac{48}{7}$, and $e = \frac{36}{7}$.

Problem LIX.

729. To find two Numbers from their Difference $= d$ given, the Sum of whose Squares shall be a Square Number yy.

Effection.

Put	1	$a =$ the one Number required.
Then	2	$a + d =$ the other by the Question.
$1 ⊙^2$	3	$aa =$ the Square of the one.
$2 ⊙^2$	4	$aa + 2ad + dd =$ the Square of the other.
$3 + 4$	5	$2aa + 2ad + dd = yy$ by the Question.
Make	6	$ra - d = y$. $r^2 > 2$
$6 ⊙^2$	7	$rra^2 - 2rda - dd = yy$
5, 7,	8	$2aa + 2ad + dd = rraa - 2rad + dd$
$8 - dd$	9	$2aa + 2ad = rraa - 2rad$.
$9 \div a$	10	$2a + 2d = rra - 2rd$.
Whence	11	$a = d \times \frac{2 + 2r}{rr - 2}$. $a + d = dr \times \frac{r + 2}{rr - 2}$ the other Number.

Ex. gr. If $d = 2$, $r = 3$. $a = \frac{16}{7}$. $e = \frac{30}{7}$.

Problem. LX.

730. To divide a given Number s (of which $3s$ equals a Square Number qq) into two such Parts a, e, that the Sum of the Cubes of those Parts shall be a Square Number $yy = m$. (In. 551.)

Effection.

Assume	1	$\frac{y}{q} - r = \overline{\frac{yy}{qq} - \frac{ss}{12}}^{\frac{1}{2}}$ or $\overline{\frac{m}{3s} - \frac{ss}{12}}^{\frac{1}{2}}$ $\begin{cases} 12rr < ss + 12rs \\ 12rr > ss - 12rs \end{cases}$
Whence	2	$y = q \times \frac{12rr + ss}{24r}$ (In. 711.)
∴	3	$a = \frac{ss - 12rr}{24r} + \frac{s}{2}$ } In. 551.
	4	$e = \frac{s}{2} - \frac{ss - 12rr}{24r}$

Ex. gr. If $s = 12$, and $r = 2$, then $a = 8$ and $e = 4$.

PROBLEM LXI.

731. To find two Numbers a, e, from their Difference given d, so that $3d$ may be a Square Number qq, the Difference of whose Cubes shall be a Square Number yy.

Effection.

Assume | 1 | $\frac{y}{q} - r = \overline{\frac{yy}{qq} - \frac{dd}{12}}^{\frac{1}{2}}$ or $\overline{\frac{n}{3d} - \frac{dd}{12}}^{\frac{1}{2}}$ In. 558.)

Whence | 2 | $y = q \times \frac{12rr + dd}{24r}$

∴ | 3 | $a = \frac{dd - 12rr}{24r} + \frac{d}{2}$ } In. 558.

| 4 | $e = \frac{dd - 12rr}{24r} - \frac{d}{2}$

∴ $12r^2 < dd - 12dr$, ∴ $d^2 > 12dr$ or $d > 12r$.

Ex. gr. If $d = 12$ and $r = \frac{1}{3}$, then $a = \frac{143}{6}$, $e = \frac{71}{6}$.

PROBLEM LXII.

732. To find two Numbers a, b, whereof the former, multiplied into the Cube of the latter, will make a Square Number.

Effection.

| 1 | $abbb$ is a Square Number by the Question.

$1 \div bb$ | 2 | ab is a Square.

Make | 3 | $rb = \overline{ab}^{\frac{1}{2}}$

$3 \odot^2$ | 4 | $rrbb = ab$

$4 \div b$ | 5 | $rrb = a$

Ex. gr. If $b = 3$ and $r = 2$, then $a = 12$.

PROBLEM LXIII.

733. To divide a given Number s, into two such Parts a, e, that their Product shall be equal to a Cube Number yyy, wanting its Root y.

Effection.

| 1 | $e = b - a$ by the Question.

Then | 2 | $ba - aa = yyy + y$ by the Question.

Make | 3 | $ae - 1 = y$

$3 \odot^3$ | 4 | $a^3e^3 - 3a^2e^2 + 3ae - 1 = yyy$

4 — 3	5	$a^3e^3 - 3a^2e^2 + 2ae = yyy - y = sa - aa$
$5 \div a$	6	$a^2e^3 - 3ae^2 + 2e = s - a$
$6 + a - 2e$	7	$a^2e^3 - 3ae^2 + a = s - 2e$
Assume	8	$s = 2e$ and so the Question becomes determinate.
$7 - \overline{s - 2e}$	9	$e^3a^2 = 3ae^2 - a$
$9 \div a$	10	$e^3a = 3e^2 - 1$
$10 \div eee$	11	$a = \frac{3e^2 - 1}{e^3} = \frac{6ss - 8}{sss}$ *Wolf.*

Ex. gr. If $s = 6$, then $a = \frac{26}{27}$, and $e = s - a = \frac{136}{27}$.

Problem LXIV.

734. To find two rational Cube Numbers $e^3 > y^3$ of the same Difference with two given Cubes Numbers $b^3 > d^3$. Whereof $2d^3 > b^3$.

Effection.

Make	1	$\frac{bb}{dd}a - d = e$
And	2	$a - b = y$
$1 \odot^3$	3	$\frac{b^6}{d^6}a^3 - \frac{3b^4}{d^3}a^2 + 3b^2a - d^3 = e^3$
$2 \odot^3$	4	$a^3 - 3a^2b^2 + 3ab^2 - b^3 = y^3$
3 — 4	5	$\frac{b^6 - d^6}{d^6}a^3 - \frac{3b^4 - 3bd^3}{d^3}a^2 + b^3 - d^3 = e^3 - y^3 = b^3 - d^3$ (by the Question.
$5 - \overline{b^3 - d^3}$	6	$\frac{b^6 - d^6}{d^6}a^3 = \frac{3b^4 - 3bd^3}{d^3}a^2$
$6 \times \frac{d^6}{a^2}$	7	$\overline{b^6 - d^6} \times a = 3b^4d^3 - 3bd^6$
$7 \div \overline{b^6 - d^6}$	8	$a = \frac{3b^4d^3 - 3bd^6}{b^6 - d^6} = \frac{3bd^3}{b^3 + d^3}$ by dividing by $b^3 - d^3$
$8 - b$	9	$a - b = \frac{2bd^3 - b^4}{b^3 + d^3} = y$
$8 \times \frac{bb}{dd} : - d$	10	$\frac{bb}{dd}a - d = \frac{2db^3 - d^4}{b^3 + d^3} = e$

Ex. gr. If $b = 5$ and $d = 4$; then $y = \frac{5}{63}$ and $e = \frac{248}{63}$.

Scholium IX.

735. *Note*, the Cubes found by the foregoing Method, are always less than the given Cubes, *i. e.* b is $> e$, and $d > y$: Which may be thus demonstrated.

Iw

	1	$2d^3$ is $> b^3$ by the Question.
$1\, w^3$	2	$2^{\frac{1}{3}}d > b$ (In. 146.)
Then	3	$2d > 2^{\frac{1}{3}}d$ (In 22.)
$\therefore$	4	$2d > b$.
$4 \div 2$	5	$d > \frac{b}{2}$
	6	$a = 3b - \frac{3b^4}{d^3+b^3}$ (Step. 8th in the last.)
	7	$b^3 > d^3$ by the Question.
$\therefore$	8	$\frac{3b^4}{2b^3} = 1\frac{1}{2}b < \frac{3b^3}{d^3+b^3}$
Consequently	9	$3b - 1\frac{1}{2}b = 1\frac{1}{2}b > a$ $b = b$
$9 - b$	10	$\frac{b}{2} > a - b = y$ Step 9. in the last.
$\therefore 5$	11	$d > y$

That is, The lesser Root given is always greater than the lesser Root sought: And consequently, since their Difference is the same, therefore the greater Root given is also greater than the Root sought. *Q. E. D.*

Problem LXV.

736. To find two rational Cubes, $e^3 > y^3$, whose Sum is equal to the Difference between two given Cubes $b^3 > d^3$, whereof $2d^3$ is $< b^3$.

Effection.

Make	1	$\frac{bb}{dd}a - d = e$
	2	$b - a = y$
$1 \odot^3$	3	$\frac{b^6}{d^6}a^3 - \frac{3b^4}{d^3}a^2 + 3b^2a - d^3 = eee$
$2 \odot^3$	4	$-a^3 + 3ba^2 - 3b^2a + b^3 = yyy$
$3+4$	5	$\frac{b^6-d^6}{d^6}a^3 - \frac{3b^4-3bd^3}{d^3}a^2 + b^3 - d^3 = e^3 + y^3 = b^3 - d^3$ (by the Question.
$5 - \overline{b^3 - d^3}$	6	$\frac{b^6-d^6}{d^6}a^3 = \frac{3b^4-3bd^3}{d^3}a^2$
$6 \times \frac{d^6}{a^2}$	7	$\overline{b^6-d^6} \times a = 3b^4d^3 - 3bd^6$

$7 \div \overline{b^6 - d^6}$

$$\begin{array}{r|l|l} 7 \div \overline{b^6 - d^6} & 8 & a = \dfrac{3b^4d^3 - 3bd^6}{b^6 - d^6} = \dfrac{3bd^3}{b^3 + d^3} \\ b - 8 & 9 & b - a = \dfrac{b^4 - 2bd^3}{b^3 + d^3} = y. \\ 8 \times \dfrac{bb}{dd} : - d & 10 & \dfrac{bb}{dd}\, a - d = \dfrac{2db^3 - d^4}{b^3 + d^3} \end{array}$$

Ex.gr. If $b^3 = 8$ and $d^3 = 1$, $b^3 - d^3 = 7$, then $e = \frac{5}{3}$, and $y = \frac{4}{3}$.

Corollary II.

737. Hence is learned the Effection of this Problem, *viz.* To divide any given rational Cube Number b^3 into three rational Cubes: Thus,

1. Assume any Cube d^3, whereof $2d^3 < b^3$.
2. Find two Cubes f^3 and g^3, whose Sum is equal to $b^3 - d^3$ by the last.

Then are d^3, f^3 and g^3, the three Cubes required, *i. e.* $d^3 + f^3 + g^3 = b^3$. *Q. E. E.*

Ex. gr. If $b^3 = 8$ and $d^3 = 1$, then $f^3 = \frac{64}{27}$ and $g^3 = \frac{125}{27}$. Proof. $1 + \frac{64}{27} + \frac{125}{27} = 8$: Or, multiplying by 27, $27 + 64 + 125 = 216$. *i. e.* $3^3 + 4^3 + 5^3 = 6^3$.

Corollary III.

738. And hence it will be easy to conceive how any given Cube b^3 may be divided into any odd Number of Cubes. For if b^3 be divided into $d^3 + f^3 + g^3$ (In. 737.) and g^3 into $h^3 + k^3 + l^3$ by the same Means; then $b^3 = d^3 + f^3 + h^3 + k^3 + l^3$, five Cubes: And again, by dividing any one of these *Ex. gr.* f^3 into $m^3 + n^3 + p^3$, we shall have $b^3 = d^3 + h^3 + k^3 + l^3 + m^3 + n^3 + p^3$, seven Cubes: And so we may proceed, at pleasure, to divide b^3 into nine, eleven, thirteen, &c. Cubes.

Scholium X.

739. *Note*, to divide a given Rational Cube into two Rational Cubes is impossible; as is demonstrated by the sagacious Dr. *Wallis*.

Problem LXVI.

740. To find two Cube Numbers $e^3 > y^3$, whose Difference shall be equal to the Sum of two given Cubes $b^3 > d^3$.

Effection.

$$\begin{array}{r|l|l} \text{Make} & 1 & a + b = e \\ \text{And} & 2 & \dfrac{bb}{dd}\, a - d = y \end{array}$$

$1 \odot^3$	3	$a^3+3a^2b+3ab^2+b^3=e^3$
$2 \odot^3$	4	$\frac{b^6}{d^6}a^3-\frac{3b^4}{d^3}a^2+3ab^2-d^3=y^3$
$3-4$	5	$\frac{d^6-b^6}{d^6}a^3+\frac{3bd^3+3b^4}{d^3}a^2+b^3+d^3=e^3-y^3=b^3+d^3$ (by the Queſtion.
$5-b^3-d^3$	6	$\frac{d^6-b^6}{d^6}a^3+\frac{3bd^3+3b^4}{d^3}a^2=0.$
$6\times\frac{d^6}{a^2}$	7	$\overline{d^6-b^6}\times a+3bd^6+3b^4d^3=0$
$\therefore$	8	$3bd^6+3b^4d^3=\overline{b^6-d^6}\times a$
$8\div\overline{b^6-d^6}$	9	$\frac{3bd^6+3b^4d^3}{b^6-d^6}=a$
$9+b$	10	$a+b=\frac{b^4+2bd^3}{b^3-d^3}=e$
$9\times\frac{bb}{dd}.-d$	11	$\frac{bb}{dd}a-d=\frac{d^4+2db^3}{b^3-d^3}=y.$

Ex. gr. If $b^3=8$ and $d^3=1$, ſo that $b^3+d^3=9$, then will $e=\frac{20}{7}$ and $y=\frac{17}{7}$.

Corollary IV.

741. Hence is learned to find two Cubes of the ſame Difference with two given Cubes $b^3>d^3$, whereof $2d^3$ is $<b^3$.

Effection.

1. By Prob. 65. find two Cubes f^3 and g^3, ſo that $f^3+g^3=b^3-d^3$.
2. By the laſt, find two Cubes h^3 and k^3, ſo that $h^3-k^3=f^3+g^3$.

Conſquently h^3-k^3 will ſatiſfy the Queſtion, *i. e.* $h^3-k^3=b^3-d^3$ (In. 21.) *Q. E. E.*

Ex. gr. If $b^3=8$, $d^3=1$, ſo that $b^3-d^3=7$, then $f^3=\frac{125}{27}$, $g^3=\frac{64}{27}$, and $h^3=\frac{2024284625}{6128487}$, $k^3=\frac{1981385216}{6128487}$, or $h=\frac{1265}{183}$, $k=\frac{1256}{183}$.

Corollary V.

742. Hence again is learned to divide two given Cubes $b^3>d^3$ into two other Cubes: Thus

1. Find $f^3-g^3=b^3+d^3$. Prob. 66.
2. Find $h^3+k^3=f^3-g^3$ (In. 741.)

Then is $h^3+k^3=b^3+d^3$ (In. 21.) *Q. E. E.*

Ex. gr. If $b^3=27$, $d^3=1$, then will $f=\frac{87}{26}$, $g=\frac{55}{26}$, and $h=\frac{83284705}{21446828}$; $k=\frac{28340511}{21446828}$.

Corol-

COROLLARY VI.

743. Hence laftly we learn to divide the Double of any given Cube b^3, into four Cubes: Thus

1. Affume $d^3 < b^3$
2. Find $f^3 + g^3 = b^3 + d^3$ (In. 742.)
3. Find $h^3 + k^3 = b^3 - d^3$ (In. 736.)
4. Then adding thefe two Equations we have $f^3 + g^3 + h^3 + k^3 = 2b^3$. *Q.E.E.*

Ex. gr. If $b^3 = 27$ and $d^3 = 1$, then will $f^3 = \frac{253452325273412980702625}{9864820937041015055552}$, $g^3 = \frac{22762660963735440852831}{9864820937041015055552}$, $h^3 = \frac{421875}{21952}$, $k^3 = \frac{148877}{21952}$.

SCHOLIUM XI.

744. The Effections of the three laft Problems, with their Corollaries, were taken chiefly from a Manufcript now by me, the Work of one Mr. *Robert Dalrymple*, a *Scotchman*, Teacher of the Mathematics fome Years ago in *Whitehaven*.

SCHOLIUM XII.

745. Becaufe it fometimes happens that one of the required Squares or Cubes in this kind of Problems are to be limited, it therefore remains that the Learner be inftructed how this is to be performed. *Ex. gr.* Suppofe, in Problem the 47th, it were required that the fide of the leffer Square $e = \frac{rr - d}{2r}$ were required to be $> q$, a Number given.

$$\begin{array}{r|l|l}
\text{Then} & 1 & \frac{rr-d}{2r} > q. \\
1 \times 2r & 2 & rr - d > 2rq \\
2 + d + 2rq & 3 & rr - 2rq > d. \\
 & 4 & r > \overline{d + qq}^{\frac{1}{2}} + q \text{ (In. 506.)}
\end{array}$$

Therefore to make e or $\frac{rr-d}{2r} > q$, for r muft be affumed fome Number greater than $\overline{d+qq}^{\frac{1}{2}} + q$.

Again, in Prob. 50. if $e = b \times \frac{rr - 1}{rr + 1}$ be required to be $> p$ and $< q$: Then

$$\begin{array}{r|l|l}
 & 1 & \frac{brr - b}{rr + 1} > p \\
 & 2 & brr - b > prr + p \\
\text{Whence} & 3 & r > \left|\frac{b+p}{b-p}\right|^{\frac{1}{2}}
\end{array}
\qquad
\begin{array}{|l|l}
1 & \frac{brr - b}{rr + 1} < q. \\
2 & brr - b = qrr + q, \\
3 & r = \left|\frac{b+q}{b-q}\right|^{\frac{1}{2}}
\end{array}$$

Therefore

Therefore to make e or $\frac{brr-b}{rr+1}$ greater than p, and leſſer than q, for r muſt be aſſumed a Number greater than $\left|\frac{b+p}{b-p}\right|^{\frac{1}{2}}$ and leſſer than $\left|\frac{b+q}{b-q}\right|^{\frac{1}{2}}$.

And after the ſame manner you may proceed to limit any other required Square or Cube.

CHAP. IV.

Of Double and Triple Quadratic and Cubic Equalities.

Problem LXVII.

746. **T**O find a Number a, which if added to b and to c will make two Squares uu and yy. $b > c$.

Effection.

	1	$a+b=uu$ } by the Queſtion.
	2	$a+c=yy$ } by the Queſtion.
Make	3	$\overline{a+b}\vert^{\frac{1}{2}}=u=b-e$
3ⓐ^2	4	$a+b=bb-2be+ee$
$4-b$	5	$a=bb-2be+ee-b$
$5+c$	6	$a+c=yy=bb-2be+ee-b+c$
Make	7	$e-c=y$
7ⓐ^2	8	$e^2-2ec+cc=yy=bb-2be+ee-b+c$ (Step 6.)
Whence	9	$e=\frac{bb-b-cc+c}{2b-2c}=\frac{b+c-1}{2}$
$\therefore$	10	$a=bb-2be+ee-b=\frac{\overline{b-c}^2-2\times\overline{b+c}+1}{4}$

Ex. gr. If $b=11$ and $c=2$, then $a=14$. Proof. $14+11=5^2$, $14+2=4^2$.

Problem. LXVIII.

747. To find a Number a, which if taken from b and from c, will leave two Squares uu, yy.

Effection.

	1	$b-a=uu$	by the Question.
	2	$c-a=yy$	
$1-2$	3	$b-c=uu-yy=q$	
$3+yy$	4	$q+yy=uu$	
Assume	5	$x<q^{\frac{1}{2}}$	
Make	6	$x+y=u$	
$6\odot^{2}$	7	$xx+2xy+yy=uu=q+yy$	(Step 4.)
Whence	8	$y=\dfrac{q-xx}{2x}$	$\therefore q>xx$ or $x<q^{\frac{1}{2}}$
$8\odot^{2}$	9	$yy=\left\|\dfrac{q-xx}{2x}\right\|^{2}=c-a$	(Step. 2.)
Whence	10	$a=c-\left\|\dfrac{q-xx}{2x}\right\|^{2}$	

Ex. gr. If $b=35$, $c=26$, and consequently $q=9$; then if $x=1$, a will $=10$. Proof. $35-10=5^{2}$. $26-10=4^{2}$.

Problem LXIX.

748. To find a Number a, from which, if b and c be taken, there will remain two Squares ee and yy. $>c$.

Effection.

	1	$a-b=ee$	by the Question.
	2	$a-c=yy$	
$2-1$	3	$b-c=yy-ee=q$	
$3+ee$	4	$b+ee=yy$	
Make	5	$x+e=y$	
$5\odot^{2}$	6	$x^{2}+2xe+ee=q+ee=yy$	(Step. 4.)
Whence	7	$e=\dfrac{q-xx}{2x}$	$: x<q^{\frac{1}{2}}$
$7\odot^{2}$	8	$ee=\left\|\dfrac{q-xx}{2x}\right\|^{2}=a-b$	(Step. 1.)
$8+b$	9	$b+\left\|\dfrac{q-xx}{2x}\right\|^{2}=a$	

Ex. gr. If $b=14$, and $c=5$, whence $q=9$, then if $x=1$; $a=30$.

PROBLEM LXX.

749. To find a Number a, which (if b be added to it, and c taken from it) will make two Squares ee, yy.

Effection.

	1	$\left.\begin{array}{l} a+b=ee \\ a-c=yy \end{array}\right\}$ by the Question.
	2	
$1-2$	3	$b+c=ee-yy=q$
$3+yy$	4	$q+yy=ee$
Make	5	$x+y=e$
$5\odot^2$	6	$xx+2xy+yy=ee=q+yy$ (Step. 4.)
Whence	7	$y=\frac{q-xx}{2x}$ $\quad x<q^{\frac{1}{2}}$
$7\odot^2$	8	$yy=\left(\frac{q-xx}{2x}\right)^2=a-c$ (Step. 2.)
$8+c$	9	$c+\left(\frac{q-xx}{2x}\right)^2=a.$

Ex. gr. If $b=12$, $c=8$, $q=b+c=20$, $x=2$, then $a=24$.

PROBLEM LXXI.

750. To find a Number a, which, if added to b and taken from c, will make two Squares ee, yy. $b+c$ being a Square.

Effection.

	1	$\left.\begin{array}{l} b+a=ee \\ c-a=yy \end{array}\right\}$ by the Question.
	2	
$1+2$	3	$b+c=ee+yy=qq$
$3-yy$	4	$qq-yy=ee$
Make	5	$xy-q=e$
$5\odot^2$	6	$x^2y^2-2qxy+qq=ee=qq-yy$ (Step. 4.)
Whence	7	$y=\frac{2qx}{x^2+1}$
$7\odot^2$	8	$yy=\frac{4qqxx}{x^4+2x^2+1}=c-a$ (Step. 2.)
Whence	9	$a=c-\left(\frac{2qx}{xx+1}\right)^2$

Ex. gr. If $b=6$, $c=3$, $qq=b+c=9$, $x=5$, then $a=\frac{282}{169}$.

PROBLEM LXXII.

751. To find a Number a, which, if multiplied into b, and into c, will produce two Squares ee, yy.

Effection.

	1	$\left.\begin{array}{l} ab=ee \\ ac=yy \end{array}\right\}$ by the Queſtion.
	2	
$1\div2$	3	$\frac{b}{c}=\frac{ee}{yy}$, therefore $\frac{b}{c}$ muſt be a Square.
$3\times yy$	4	$\frac{yyb}{c}=ee$
Make	5	$yx=ee$
4, 5,	6	$\frac{yyb}{c}=yx$
	7	$y=\frac{cx}{b}$
$7\odot^{2}$	8	$yy=\frac{ccxx}{bb}=ac$ (Step. 1.)
$8\div c$	9	$\frac{cxx}{bb}=a$.

Ex. gr. If $b=8$, $c=2$, $x=3$, then $a=\frac{18}{64}$. Whence $ab=\frac{144}{64}$ and $ac=\frac{36}{64}$.

PROBLEM LXXIII.

752. To find a Number a, which, if divided by b and by c, will make two Squares ee, yy.

Effection.

	1	$\left.\begin{array}{l} \frac{a}{b}=ee \\ \frac{a}{c}=yy \end{array}\right\}$ by the Queſtion.
	2	
1, 2.	3	$a=bee=cyy$
$3\div c$	4	$\frac{b}{c}ee=yy$, therefore $\frac{b}{c}$ muſt be a Square.
Make	5	$xe=yy$
4, 5,	6	$e=\frac{cx}{b}$

$6\odot^2$	7	$ee = \frac{ccxx}{bb} = \frac{a}{b}$ (Step. 1.)
$7 \times b$	8	$\frac{ccxx}{b} = a$

Ex. gr. If $b=8$, $c=2$ and $x=3$, then $a=\frac{36}{8}=\frac{9}{2}$.

Problem LXXIV.

753. To find a Number a, which, if multiplied into b, and divided by c, will make two Squares ee, yy.

Effection.

	1	$ab=ee$ } by the Question.
	2	$\frac{a}{c}=yy$ }
1, 2,	3	$a=\frac{ee}{b}=cyy$
$3 \times b$	4	$ee=bcyy$ $\therefore$ bc must be a Square.
Make	5	$xy=bcyy$
Whence	6	$\frac{xx}{bbcc}=yy=\frac{a}{c}$ (Step. 2.)
$6 \times c$	7	$\frac{xx}{bbc}=a$.

Ex. gr. If $b=8$, $c=2$, $x=12$; $a=\frac{144}{32}$ or $\frac{36}{8}$ or $\frac{9}{2}$. Proof, $ab=\frac{72}{2}=36$. $\frac{a}{c}=\frac{9}{4}$.

Problem LXXV.

754. To find a Number a, which, if b be added to it, and subtracted from it, the Sum and Difference will be two Squares ee, yy.

Effection.

	1	$a+b=ee$ } by the Question.
	2	$a-b=yy$ }
1—2	3	$2b=ee-yy$
$3+yy$	4	$2b+yy=ee$
Make	5	$x+y=e$ ($x<\overline{2b}^{\frac{1}{2}}$ Step. 7.)
4, 5.	6	$2b+yy=xx+2xy+yy$

Whence

Whence	7	$\left(\frac{2b-xx}{2x}\right)^2 = yy = a-b$ (Step. 2.)
$7+b$	8	$\left(\frac{2b-xx}{2x}\right)^2 + b = a$.

Ex. gr. If $b=8$, and $x=2$, then $a=17$. Proof. $a+b=25$, $a-b=9$.

Problem LXXVI.

755. To find a Number a, which, if added to and subtracted from its Square, will make two Squares ee, yy.

Effection.

	1	$aa+a=ee$ } by the Question.
	2	$aa-a=yy$ }
Make	3	$a^2-a=\overline{a-u}^2=a^2-2au+uu=yy$
Whence	4	$a=\frac{uu}{2u-1}$
$4 \odot^2$	5	$aa=\frac{u^4}{4u^2-4u+1}$
$5+4$	6	$aa+a=\overline{u^2+2u-1}\times\left(\frac{u}{2u-1}\right)^2=ee$ (Step. 1.) Therefore $uu+2u-1$ is a Square (In. 156.)
Make	7	$uu+2u-1=\overline{z+u}^2=zz+2zu+uu$
Whence	8	$u=\frac{zz+1}{2-2z}$ $\therefore$ $z<1$
4, 8,	9	$a=\frac{uu}{2u-1}$

Ex. gr. If $z=\frac{1}{2}$ then $a=\frac{25}{24}$.

Problem. LXXVII.

756. To find two Numbers, the greater of which is to the lesser as b to c; and if each be added to the Square of their Sum, the Sums will make two Squares ee, yy.

Effection.

Put $ba>ca$ for the two Numbers required, and make $b+c=s$ or $ba+ca=sa$.

	1	$s^2a^2+ba=ee$ } by the Queſtion.
	2	$s^2a^2+ca=yy$ }
Put	3	$s^2a^2+ba=\overline{sa+u}^2=s^2a^2+2sau+u^2=ee$
Whence	4	$a=\frac{uu}{b-2su}$
And	5	$s^2a^2=\frac{ssu^4}{\overline{b-2su}^2}$
5+4	6	$s^2a^2+ca=\overline{s^2u^2-2scu+bc}\times\overline{\frac{u}{b-2su}}^2=yy$ (Step. 2.)
		Therefore $ssuu-2scu+bc$ muſt be a Square (In. 156.)
Make	7	$s^2u^2-2scu+bc=\overline{su-z}^2=s^2u^2-2szu+zz$
Whence	8	$u=\frac{zz-bc}{2s\times z-c}$ $\therefore$ $zz>bc$ and $z>c$
4, 8,	9	$a=\frac{uu}{b-2su}$

Ex. gr. If $b=3$, $c=1$, $s=4$, $z=2$, then $u=\frac{1}{8}$, $a=\frac{1}{128}$ $\therefore$ conſequently, $ba=\frac{3}{128}$, $ca=\frac{1}{128}$, whoſe Sum $sa=\frac{4}{128}$ or $\frac{1}{32}$.

Problem LXV.

757. To find two Numbers, the greater of which is to the leſſer as b to c; and if the Square of each be added to their Sum, the Sums will be two Square ee, yy.

Effection.

Put $ba>ca$ for the two Numbers required, and make $b+c=s$ or $ba+ca=sa$.

	1	$b^2a^2+sa=e^2$ } by the Queſtion.
	2	$c^2a^2+sa=y^2$ }
Put	3	$b^2a^2+sa=\overline{ba+u}^2$ or $b^2a^2+2bau+u^2=e^2$
Whence	4	$sa=\frac{suu}{s-2bu}$
And	5	$ccaa=\frac{ccu^4}{\overline{s-2bu}^2}$
5+4	6	$c^2a^2+sa=\overline{c^2u^2-2sbu+s^2}\times\overline{\frac{u}{s-2bu}}^2=yy$ (Step. 2.)
		$\therefore$ $c^2u^2-2sbu+ss$ muſt be a Square (In. 156.)

Make

Make | 7 | $c^2u^2-2sbu+ss=\overline{cu-z}^2$ or $c^2u^2-2cuz+zz$
Whence | 8 | $u=\dfrac{ss-zz}{2sb-2cz}$ $\therefore z<s$.
4, 8, | 9 | $a=\dfrac{uu}{s-2bu}=\dfrac{\overline{ss-zz}^2}{4zx\overline{sz}\times\overline{\overline{bb+cc}-bc}\times\overline{ss+zz}}$

Ex. gr. If $b=5$, $c=1$, $s=6$ and $z=2$, then $a=\frac{8}{7}$ or $\frac{56}{49}$; consequently $ba=\frac{40}{7}$ and $ca=\frac{8}{7}$, whose Sum $sa=\frac{48}{7}$ or $\frac{336}{49}$.

Scholium XIII.

759. The three next Problems are *Triple Equalities*, but are reduced to *Double* ones, by the Representation of the Terms. And for their more easy Effection, the following Lemma is necessary to be premised.

Lemma.

759. If d be put to represent the Difference between any two Squares $aa>ee$, and r be put for any other Number at pleasure, whose Square rr is greater than d, then $\dfrac{rr+d}{2r}$ will be the Side of the greater Square sought or a; and $\dfrac{rr-d}{2r}$ the side of the lesser or e (In. 714.) therefore if $rc=d$, then $\dfrac{rr+rc}{2r}$ or $\dfrac{r+c}{2}=a$, and $\dfrac{rr-rc}{2r}$ or $\dfrac{r-c}{2}=e$. Whence

If rc represents the Difference between any two Squares, I say Half the Sum of those *Factors*, *viz.* $\dfrac{r+c}{2}$ will be the side of the greater Square; and half their Difference, *viz.* $\dfrac{r-c}{2}$ the Side of the lesser.

Problem LXXIX.

760. To find three such Numbers, that if the Product of every two of them be added to the third, the three Sums will be so many Squares, aa, ee, yy.

Effection.

Put a for the first Number sought, $a-bb$ for the second, and abb for the third, by which Means the Sum of the Product of the first and second, added to the third, is equal to the Square of a or aa by Hypothesis.

1—2

	1	$aabb - ab^4 + a = ee$ } by the Question.
	2	$aabb + a - bb = yy$ }
1—2	3	$bb - ab^4 = ee - yy = d$ (In. 714.)
Alſo	4	$bb - ab^4 = \frac{bbb}{2} \times \overline{\frac{2}{b} - 2ab} = bb \times \overline{1 - abb} = b \times \overline{b - abb} = rc$ by the Lemma. But the firſt of theſe Pairs of Factors muſt be taken, *viz.* $\frac{bbb}{2}$ and $\frac{2}{b} - 2ab$, becauſe of $2ab$, (In. 711.)
∴	5	$\frac{2}{b} - 2ab = r$ and $\frac{bbb}{2} = c$
$\frac{r+c}{2} = e$ } by the Lemma.	6	$\frac{1}{b} - ab + \frac{bbb}{2} = e$ } Either of which may be involved and equated to its proper Square in the 1ſt or 2d Step.
$\frac{r-c}{2} = y$ }	7	$\frac{1}{b} - ab - \frac{bbb}{4} = y$ }
1, 6.	8	$aabb - ab^4 + a = \frac{1}{bb} + \frac{bb}{2} - 2a + \frac{b^6}{16} - \frac{ab^4}{2} + a^2b^2$
Whence	9	$a = \frac{b^6 + 8b^2 + \frac{16}{bb}}{48 - 8b^4}$ ∴ $6 > b^4$ or rather $1 > b$
	10	$a - bb = \frac{9b^6 - 4bb + \frac{16}{bb}}{48 - 8b^4}$ the ſecond Number required.
	11	$abb = \frac{b^8 + 8b^4 + 16}{48 - 8b^4}$ the third.

Ex. gr. If $b = \frac{1}{2}$, then $a = \frac{16900}{12160}$. $a - bb = \frac{13860}{12160}$. $abb = \frac{4225}{12160}$.

Problem LXXX.

761. To find three ſuch Numbers, that if the Product of every two of them be leſſened by the third, the three Remainders will be ſo many Squares aa, ee, yy.

Effection.

Put a for the firſt Number ſought, $a + bb$ for the ſecond, and abb for the third.

	1	$aa + abb - abb = aa$ } by the Queſtion.
	2	$aabb - a - bb = ee$ }
	3	$aabb + abbbb - a = yy$ }
3—2	4	$ab^4 + bb = yy - ee = d = \frac{b^3}{2} \times \overline{2ab + \frac{2}{b}}$

∴	5	$ab+\frac{1}{b}+\frac{b^3}{4}=\frac{r+c}{2}=y$	In. 760.
And	6	$ab+\frac{1}{b}-\frac{b^3}{4}=\frac{r-c}{2}=e$	
3, 5.	7	$a^2b^2+ab^4-a=a^2b^2+\frac{ab^4}{2}+2a+\frac{b^6}{16}+\frac{b^2}{2}+\frac{1}{bb}$.	
Whence	8	$a=\frac{b^6+8b^2+\frac{16}{bb}}{8b^4-48}$ ∴ $b>1$	

Ex. gr. If $b=2$, then $a=\frac{5}{4}$, $a+bb=\frac{21}{4}$, $abb=\frac{20}{4}$.

PROBLEM LXXXI.

762. To find three such Numbers, that if to the Square of every one of them the Sum of the other two be added, the three Sums will be so many Squares.

Effection.

For the three Numbers sought put a, $2a$, 1.

	1	$aa+2a+1=\overline{a+1}^2$	by the Question.
	2	$4aa+a+1=ee$	
	3	$3a+1=yy$	
2—3	4	$4aa-2a=ee-yy=2\times\overline{2aa-a}$ or $a\times\overline{4a-2}=rc=d$	
Make	5	$r=4a-2$ and $c=a$ (In. 718.)	
∴	6	$\frac{5}{2}a-1=\frac{r+1}{2}=e$	In. 760.
	7	$\frac{3}{2}a-1=\frac{r-c}{2}=y$	
2, 6	8	$4aa+a+1=\frac{25}{4}aa-5a+1$	
Whence	9	$a=\frac{8}{3}$, consequently $2a=\frac{16}{3}$	

The three Numbers then are 1, $\frac{8}{3}$ and $\frac{16}{3}$.

PROBLEM LXXXII.

763. To find three Numbers, which, by the Addition of each to the Square of their Sum will make as many Squares.

Effection.

Put a for the Sum of the three Numbers required, and assume $b>c>d$, any three Numbers, whose Sum is $<\frac{1}{2}$: Then make $2ab+bb=$ the first Number; $2ac+cc=$ the second; and $2ad+dd=$ the third.

Q Then

$$\begin{array}{l|l|l}
\text{Then} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} 1 \\ 2 \\ 3 \end{array} & \left. \begin{array}{l} aa+2ab+bb=\overline{a+b}^2 \\ aa+2ac+cc=\overline{a+c}^2 \\ aa+2ad+dd=\overline{a+d}^2 \end{array} \right\} \text{by Hypothesis.} \\
 & 4 & 2ab+2ac+2ad+bb+cc+dd=a \text{ by the Question.} \\
\text{Whence} & 5 & a=\dfrac{bb+cc+dd}{1-2\times\overline{b+c+d}} \quad \therefore 1>2\,\overline{b+c+d}
\end{array}$$

Ex. gr. If $b=\frac{1}{4}$, $c=\frac{1}{8}$, $d=\frac{1}{16}$, then $a=\frac{21}{32}$; whence $2ab+bb=\frac{25}{64}$. $2ac+cc=\frac{23}{128}$. $2ad+dd=\frac{22}{256}$. Proof, $\frac{25}{64}+\frac{23}{128}+\frac{22}{256}=\frac{21}{32}=a$.

PROBLEM. LXXXIII.

764. To find three Numbers, which will leave as many Squares, after the Subtraction of each from the Square of their Sum.

Effection.

Put a for the Sum of the three Numbers required, and assume $b>c>d$, so that $b+c+d$ be $>\frac{1}{2}$ and <1: Next make $2ab-bb=$ the first Number; $2ac-cc=$ the second; $2ad-dd=$ the Third. Then

$$\begin{array}{l|l|l}
 & 1 & 2a\times\overline{b+c+d}-bb-cc-dd=a \text{ by the Question.} \\
\text{Whence} & 2 & a=\dfrac{bb+cc+dd}{2\times\overline{b+c+d}-1}
\end{array}$$

Ex. gr. If $b=\frac{1}{2}$, $c=\frac{1}{4}$ $d=\frac{1}{8}$, then $a=\frac{7}{16}=\frac{14}{32}$. $2ab-bb=\frac{6}{32}$. $2ac-cc=\frac{5}{32}$. $2ad-dd=\frac{3}{32}$.

PROBLEM LXXXIV.

765. To find a *Non Quadrate* (*i. e.* a not Square) Number E, which being added to three *Non Quadrates* A, B, C, will make as many Squares.

Effection.

$$\begin{array}{r|l|l}
\text{Put} & 1 & 2ae+aa=E \text{ the first Number sought} \\
\text{And} & 2 & ee-4ae=A \text{ also sought} \\
\text{Then} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} 3 \\ 4 \\ 5 \end{array} & \left. \begin{array}{l} ee-2ae+aa=\overline{e-a}^2=A+E \\ B+2ae+aa=yy=B+E \\ C+2ae+aa=uu=C+E \end{array} \right\} \text{by the Question.} \\
\text{Make} & 6 & B-a=y \\
6,\ 4, & 7 & BB-2Ba+aa=B+2ae+aa \\
\text{Whence} & 8 & a=\dfrac{BB-B}{2B+2e} \\
\text{Make} & 9 & C+a=u
\end{array}$$

Whence

$$\begin{array}{r|r|l}
\text{Whence} & 10 & a=\dfrac{CC-C}{2e-2C} \\
8,\ 10 & 11 & \dfrac{BB-B}{B+e}=\dfrac{CC-C}{e-C} \\
\therefore & 12 & e=\dfrac{BC\times\overline{B+C-2}}{BB+C-CC-B}
\end{array}$$

Ex. gr. If $B=3$ and $C=2$, then $e=\frac{9}{2}$, and $a=\frac{2}{5}$; consequently $A=ee-4ae=\frac{1305}{100}$. $E=2ae+aa=\frac{376}{100}$. And if we multiply all by 100, we will have $A=1305$, B$=300$, $C=200$, and E$=376$ in Integers. Proof, $A+E=1681=\overline{41}^2$. $B+E=676=\overline{26}^2$. $C+E=576=\overline{24}^2$.

Problem LXXXV.

766. To find a Number A, to which, if one given Cube b^3 be added, and another lesser given Cube d^3 subtracted from it, that Sum and Difference will be two Cubes e^3, y^3.

Effection.

$$\begin{array}{r|r|l}
\text{Put}\left\{\begin{array}{l}\\ \\ \end{array}\right. & \begin{array}{r}1\\2\end{array} & \begin{array}{l} bbb+3d^2a+\dfrac{3d^4}{b^3}a^2+\dfrac{d^6}{b^6}a^3=\overline{b+\dfrac{dd}{bb}a}^3=b^3+A=e^3 \\ a^3-3a^2d+3ad^2-d^3=A-d^3=y^3 \end{array} \\
\text{Then} & 3 & a^3-3a^2d+3ad^2=3ad^2+\dfrac{3d^4}{b^3}a^2+\dfrac{d^6}{b^6}a^3=A \\
\text{Whence} & 4 & a=\dfrac{3b^3d}{b^3-d^3}.
\end{array}$$

Ex. gr. If $b^3=8$ and $d=1$, then $a=\frac{24}{7}$, consequently $A=\frac{5256}{343}$. And, if we multiply all by 343, we will have $b^3=2744=14^3$, $d^3=343=7^3$, and $A=5256$. Proof $A+b^3=8000=\overline{20}^3$ and $A-d^3=4913=17^3$.

Problem LXXXVI.

767. To find three Numbers, which will leave as many Cubes after the Subtraction of each from the Cube of their Sum.

Effection.

For the three Numbers put a, b, c, whose Sum is d, and for the three Cubes remaining put e^3, f^3, g^3.

$a=$?

$a=?$	1	$a+b+c=d$
$b=?$	2	$d^3-a=e^3$
$c=?$	3	$d^3-b=f^3$ by the Question.
$d=?$	4	$d^3-c=g^3$
$e=?$	5	(*)
$f=?$	6	(*)
$g=?$	7	(*)
d^3-2	8	$a=d^3-e^3$
d^3-3	9	$b=d^3-f^3$
d^3-4	10	$c=d^3-g^3$
$n=?$	11	Let $e=p-n$
$p=?$	12	Let $f=4n-p$
5 (*)	13	Let $g=2n$
11 $\odot^3$	14	$e^3=+p^3-3p^2n+3pn^2-n^3$
12 $\odot^3$	15	$f^3=-p^3+12p^2n-48pn^2+64n^3$
13 $\odot^3$	16	$g^3=8n^3$
6 (*)	17	Let $d=4n$
17 $\odot^3$	18	$d^3=64n^3$
18—14	19	$d^3-e^3=65n^3-3pn^2+3p^2n-p^3=a$
18—15	20	$d^3-f^3=48pn^2-12p^2n+p^3=b$
11—16	21	$d^3-g^3=56n^3=c$
19+20+21	22	$121n^3+45pn^2-9p^2n=a+b+c=d=4n$ (Step. 8, 9, 10.)
22÷n	23	$121n^2+45pn-9p^2=4$
7 (*)	24	Let $11n+p=\overline{121n^2+45pn-9p^2}^{\frac{1}{2}}=2$
23, 24	25	$121n^2+45pn-9p^2=121n^2+22pn+p^2=4.$
Whence	26	$p=\frac{23}{10}n$
11,	27	$e=p-n=\frac{13}{10}n$
17,	28	$d=4n=\frac{40}{10}n$ ∴ $d^3=\frac{64000}{1000}n^3$
12,	29	$f=4n-p=\frac{17}{10}n$ ∴ $f^3=\frac{4913}{1000}n^3$
13,	30	$g=2n=\frac{20}{10}n$ ∴ $g^3=\frac{8000}{1000}n^3$
8,	31	$a=d^3-e^3=\frac{61803}{1000}n^3$
9,	32	$b=d^3-f^3=\frac{59087}{1000}n^3$
10,	33	$c=d^3-g^3=\frac{56000}{1000}n^3$
31+32+33	34	$a+b+c=\frac{176890}{1000}n^3=\frac{17689}{100}n^3=d=4n$ (Step. 17.)
34÷n	35	$\frac{17689}{100}nn=4$
35 ω^2	36	$\frac{133}{10}n=2$
Whence	37	$n=\frac{20}{133}$
31,	38	$a=\frac{61803}{1000}n^3=\frac{494424}{2352637}$
32,	39	$b=\frac{59087}{1000}n^3=\frac{472696}{2352637}$
33,	40	$c=\frac{56000}{1000}n^3=\frac{448000}{2352637}$
∴	41	$d=\frac{80}{133},\ e=\frac{26}{133},\ f=\frac{34}{133},\ g=\frac{40}{133},\ p=\frac{46}{133}.$

For

For the Exclusion of Negatives the Doctor proceeds thus: Instead of the Equation at the 24th Step, he takes $11n+qp=+2$ or -2: And then enquires what Numbers may be the Values of q, so as to make both e and f positive.

To this purpose having borrowed the 11th, 12th, and 23d Equations, he goes on.

	11	$e=p-n$
	12	$f=4n-p$
	23	$121n^2+45np-9p^2=4$
	42	$11n+qp=+2$ or -2
$42 \odot^2$	43	$121n+22qpn+q^2p^2=4$
$23-43$	44	$45np-9pp-22qpn-q^2p^2=0$
$44 \div p$	45	$45n-9p-22qn-qqp=0$
Whence	46	$45n-22qn=qqp+9p$
$\therefore$	47	$n:p=qq+9:45-22q$
First Scope	48	$e=p-n=0$, or $y=n$
47, 48	49	$qq+9=45-22q$
$49+22q-9$	50	$qq+22q=36$
Whence	51	$q=-11+\overline{157}^{\frac{1}{2}}=+1.529964$, or $-11-\overline{157}^{\frac{1}{2}}=-23.529964$
	52	$\therefore q$ taken between $+1\frac{529}{1000}$ and $-23\frac{529}{1000}$ makes $e>0$.
Second Scope	53	$f=4n-p=0$, or $4n=p$
46,	54	$45n-22qn=4qqn+36n$, or $45-22q=4qq+36$
Whence	55	$q=+0.382491$ &c. or -5.882491 &c.
	56	$\therefore q$ between $+\frac{382}{1000}$ and $-5\frac{882}{1000}$ makes $f<0$, otherwise greater than nothing
Consequently	57	Both e and f will be >0, when you take q between $+1.529964$, &c and $+0.382491$ &c. or between -5.882491 &c. and -23.529964 &c.

Thus far Dr. *Pell*, who also proceeds by making $q=\frac{1}{2}$, for one Example, and -6 for another: And then goes to Tables, by the help whereof, great Varieties of such Answers may be readily had.

The End of the Sixth and last PART.

TRIUNI DEO GLORIA.

An

An Alphabetical INDEX *of what is contained in the foregoing* TREATISE.

Note, *The Numbers refer to the* Institutions *in the Margin.*

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Nomina-

INDEX.